Testing the Kerr paradigm

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Black holes play a key role in modern physics

Recent Nobel Prizes in astrophysics/cosmology:

1983: **Chandrasekhar**, Fowler

1993: Hulse, Taylor

2002: Davis, Koshiba, **Giacconi**

2006: Mather, Smoot

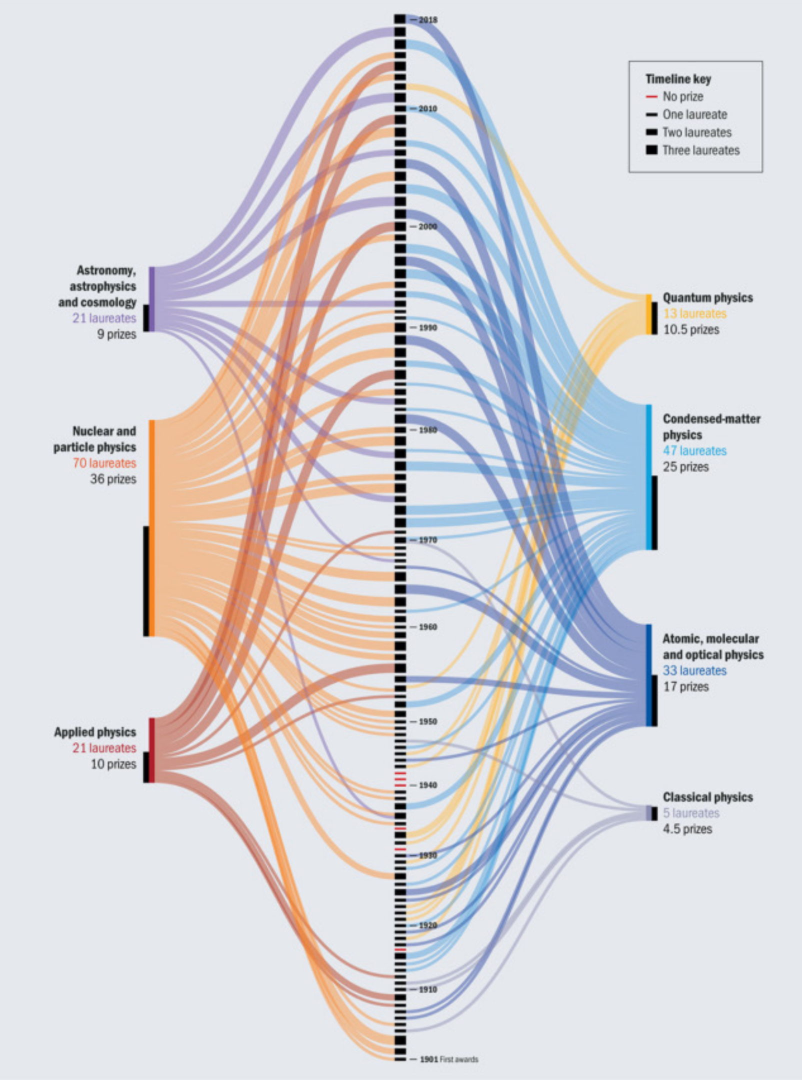
2011: Perlmutter, Schmidt, Riess

2017: **Weiss, Barish, Thorne**

2019: Peebles, Mayor, Queloz

2020: **Penrose, Genzel, Ghez**

Are they really the Kerr black holes of general relativity?



What do we know about black hole solutions beyond GR?

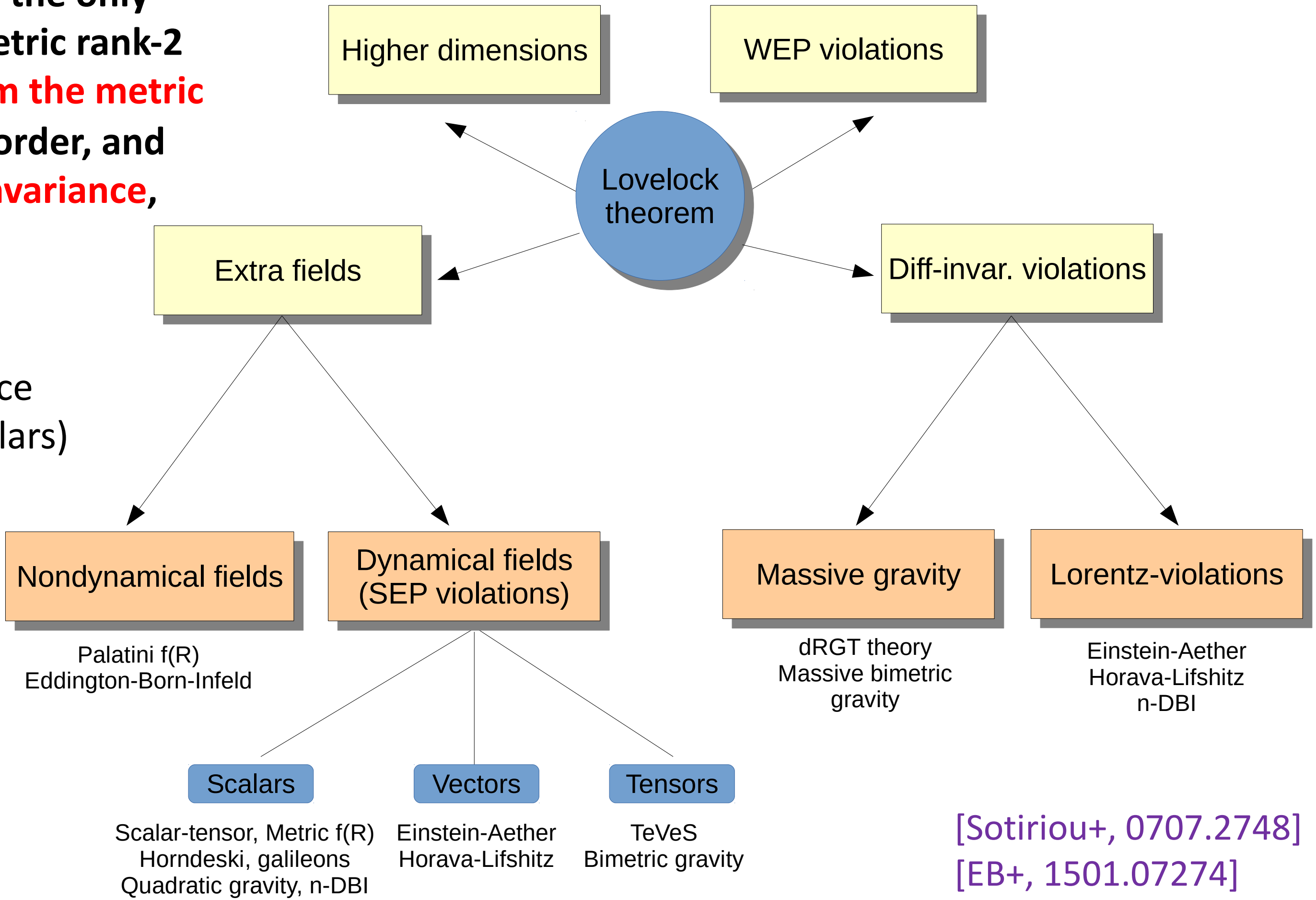
A guiding principle to modified GR: Lovelock's theorem

In four spacetime dimensions the only divergence-free (WEP) symmetric rank-2 tensor constructed solely from the metric and its derivatives up to 2nd order, and preserving diffeomorphism invariance, is the Einstein tensor plus Λ.

Generic modifications introduce additional fields (simplest: scalars)

Minimal requirements:

- Action principle
- Well-posed
- Testable predictions
- Black holes, neutron stars
- Cosmologically viable



[Sotiriou+, 0707.2748]
[EB+, 1501.07274]

(Often) black hole binaries are the same as in GR! Scalar-tensor: no-hair theorems

$$S=\frac{1}{16\pi}\int\sqrt{-g}d^4x\left[\phi R-\frac{\omega(\phi)}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}+M(\phi)\right]+\int\mathcal{L}_{\rm M}(g^{\mu\nu},\Psi)d^4x$$

Orbital period derivative: $\frac{\dot{P}}{P}=-\frac{\mu m}{r^3}\kappa_D(s_1-s_2)^2-\frac{8}{5}\frac{\mu m^2}{r^4}\kappa_1$

$$\kappa_D=2\mathcal{G}\xi\left(\frac{\omega^2-m_s^2}{\omega^2}\right)^{\frac{3}{2}}\Theta(\omega-m_s)$$

$$\kappa_1=\mathcal{G}^2\left[12-6\xi+\xi\Gamma^2\left(\frac{4\omega^2-m_s^2}{4\omega^2}\right)^{\frac{5}{2}}\Theta(2\omega-m_s)\right]$$

$$\xi=\frac{1}{2+\omega_{\rm BD}}$$

$$\mathcal{G}=1-\xi(s_1+s_2-2s_1s_2)$$

$$\Gamma=1-2\frac{s_1m_2+m_1s_2}{m}$$

For black hole binaries, $s_1=s_2=\frac{1}{2}$ and dipole vanishes identically

Quadrupole: $\Gamma=0$

Result extended to higher PN orders; it is exact in the large mass ratio limit
[Will & Zaglauer 1989; Alsing+, 1112.4903; Mirshekari & Will, 1301.4680; Yunes+, 1112.3351; Bernard 1802.10201, 1812.04169, 1906.10735]

Ways around: matter (but EOS degeneracy), cosmological BCs (but small corrections), or **curvature itself sourcing the scalar field: dCS, EsGB** [Yagi+ 1510.02152]

Systematically exploring *small* corrections: the effective field theory (EFT) viewpoint

Expand all operators in the action in terms of some length scale
(must be macroscopic to be relevant for GW tests).
Theories: sum over curvature invariants with scalar-dependent coefficients

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\left[R + \sum_{n=2}^{\infty}\ell^{2n-2}\mathcal{L}_{(n)}\right]$$

and more specifically, at order ℓ^4

EsGB dCS (dilaton+axion)

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\Big\{R + \alpha_1\phi_1\ell^2 R_{\mathrm{GB}} + \alpha_2(\phi_2\cos\theta_m + \phi_1\sin\theta_m)\ell^2 R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$$
$$+\lambda_{\mathrm{ev}}\ell^4 R_{\mu\nu}^{\rho\sigma}R_{\rho\sigma}^{\delta\gamma}R_{\delta\gamma}^{\mu\nu} + \lambda_{\mathrm{odd}}\ell^4 R_{\mu\nu}^{\rho\sigma}R_{\rho\sigma}^{\delta\gamma}\tilde{R}_{\delta\gamma}^{\mu\nu} - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2\Big\}$$

Einsteinian cubic gravity (+parity-breaking) - causality constraints [Camanho+ 1407.5597]
Next order, no new DOFs [Endlich-Gorbenko-Huang-Senatore, 1704.01590]

$$S_{(4)} = \frac{\ell^6}{16\pi G}\int d^4x\sqrt{|g|}\left\{\epsilon_1\mathcal{C}^2 + \epsilon_2\tilde{\mathcal{C}}^2 + \epsilon_3\mathcal{C}\tilde{\mathcal{C}}\right\}$$

$$\mathcal{C} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \quad \tilde{\mathcal{C}} = R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$$

[Cano-Ruipérez, 1901.01315; Cano-Fransen-Hertog, 2005.03671. See also work by Hui, Penco…]

Einstein-scalar-Gauss-Bonnet gravity: a loophole in no-hair theorems

Horndeski Lagrangian: most general scalar-tensor theory with second-order EOMs

$$S = \sum_{i=2}^{5} \int d^4x \sqrt{-g}\mathcal{L}_i$$

$$G_i = G_i(\phi, X) \qquad \phi^2_{\mu\nu} \equiv \phi_{\mu\nu}\phi^{\mu\nu}$$

$$X = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \qquad \phi^3_{\mu\nu} \equiv \phi_{\mu\nu}\phi^{\nu\alpha}\phi^\mu_\alpha$$

$$\mathcal{L}_2 = G_2$$
$$\mathcal{L}_3 = -G_3\Box\phi$$
$$\mathcal{L}_4 = G_4 R + G_{4X}\left[(\Box\phi)^2 - \phi^2_{\mu\nu}\right]$$
$$\mathcal{L}_5 = G_5 G_{\mu\nu}\phi^{\mu\nu} - \tfrac{1}{6}G_{5X}\left[(\Box\phi)^3 + 2\phi^3_{\mu\nu} - 3\phi^2_{\mu\nu}\Box\phi\right]$$

Set:

$$G_2 = X + 8f^{(4)}X^2(3 - \ln X)$$
$$G_3 = 4f^{(3)}X(7 - 3\ln X)$$
$$G_4 = \tfrac{1}{2} + 4f^{(2)}X(2 - \ln X)$$
$$G_5 = -4f^{(1)}\ln X$$

$$S = \int d^4x \sqrt{-g}\left(\frac{1}{2}R + X + f(\phi)\mathcal{G}\right)$$

$$\mathcal{G} \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

Shift symmetry: invariance under $\phi \to \phi + c$, i.e. $G_i = G_i(X)$

EsGB is **a special case of Horndeski *and* of quadratic gravity**

[Kobayashi+, 1105.5723; Sotiriou+Zhou, 1312.3622; Maselli+, 1508.03044]

Black hole spontaneous scalarization

$$\Box\varphi = -f_{,\varphi}\mathscr{G}$$

$$\int_{\mathscr{V}} \mathrm{d}^4x\sqrt{-g}\left[f_{,\varphi}\Box\varphi + f_{,\varphi}^2(\varphi)\mathscr{G}\right] = 0$$

Integrate by parts, divergence theorem:

$$\int_{\mathscr{V}} \mathrm{d}^4x\sqrt{-g}\left[f_{,\varphi\varphi}\nabla^\mu\varphi\nabla_\mu\varphi - f_{,\varphi}^2(\varphi)\mathscr{G}\right] = \int_{\partial\mathscr{V}} \mathrm{d}^3x\sqrt{|h|}f_{,\varphi}n^\mu\nabla_\mu\varphi$$

The RHS vanishes for stationary, asymptotically flat spacetimes; if $\boxed{f_{,\varphi\varphi}\mathscr{G} < 0}$ both terms on the LHS vanish separately, i.e. $\varphi = \varphi_0 = c$

In alternative, linearize the scalar field equation: $[\Box + f_{,\varphi\varphi}(\varphi_0)\mathscr{G}]\delta\varphi = 0$

$m_{\mathrm{eff}}^2 = -f_{,\varphi\varphi}\mathscr{G}$ is an **effective mass** for the perturbation – tachyonic instability

Will not discuss here – Sotiriou's talk

[Silva+, 1711.02080]

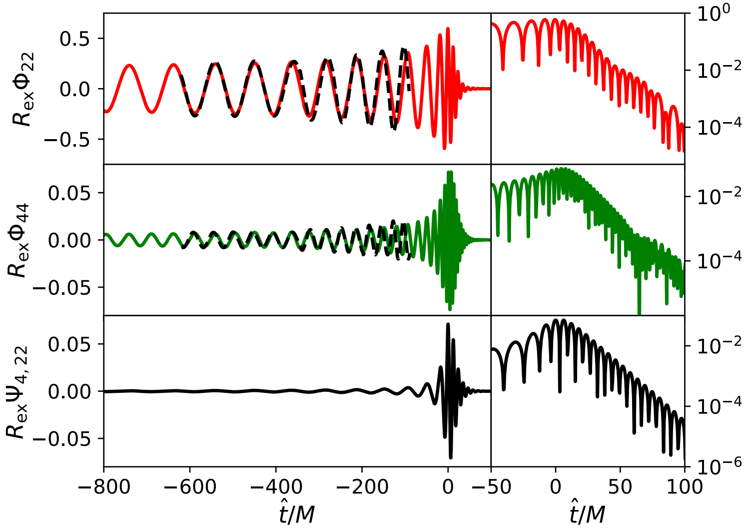
Binaries in Einstein-scalar-Gauss-Bonnet: analytical and numerical progress

- EsGB: subclass of Horndeski theory that evades no-hair theorems
  Scalarized solution exist, can be stable, can differ sensibly from GR
  Interesting phenomenology for spin-induced scalarization

- BHBs produce dipolar radiation: post-Newtonian work
  [Yagi+ 1510.02152; Julié+, 1909.05258; Shiralilou+, 2105.13972]

- Effective-one-body: work in progress
  [Bernard+, Julié+Baibhav+EB…]

- NR mergers:
  inspiral, weak coupling [Witek+ 1810.05177]
  head-on, full theory [Ripley-East 2105.08571]



- Related work also in dynamical Chern-Simons
  [Okounkova+ 1705.07924, 1906.08789, 1911.02588, 2001.03571]

Bottom line:

In many theories, black hole solutions are the same as in GR

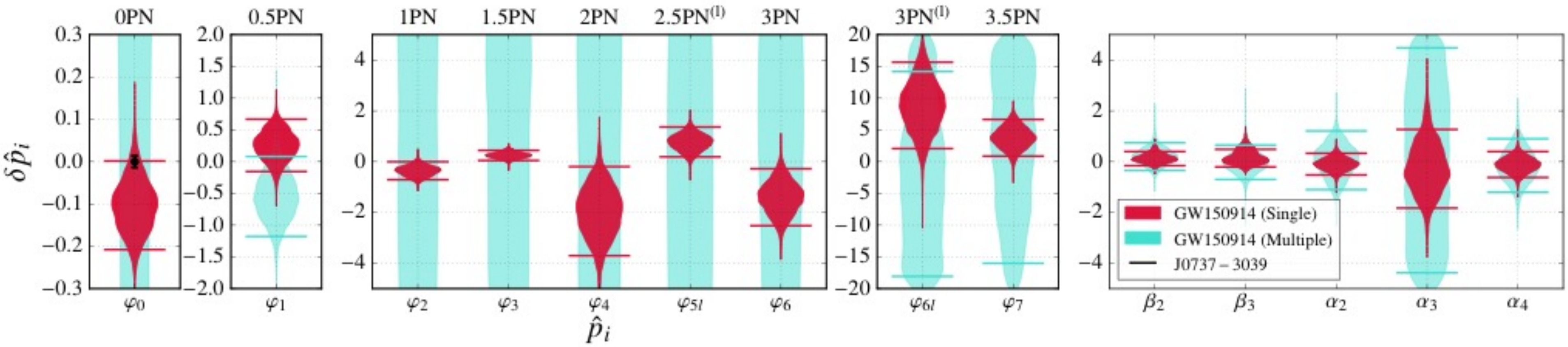
In EsGB gravity, black holes differ from GR
because of curvature/spin induced “spontaneous scalarization”
and can produce dipolar radiation

Can we test this with gravitational waves?

Parametrized post-Einstein formalism in the inspiral

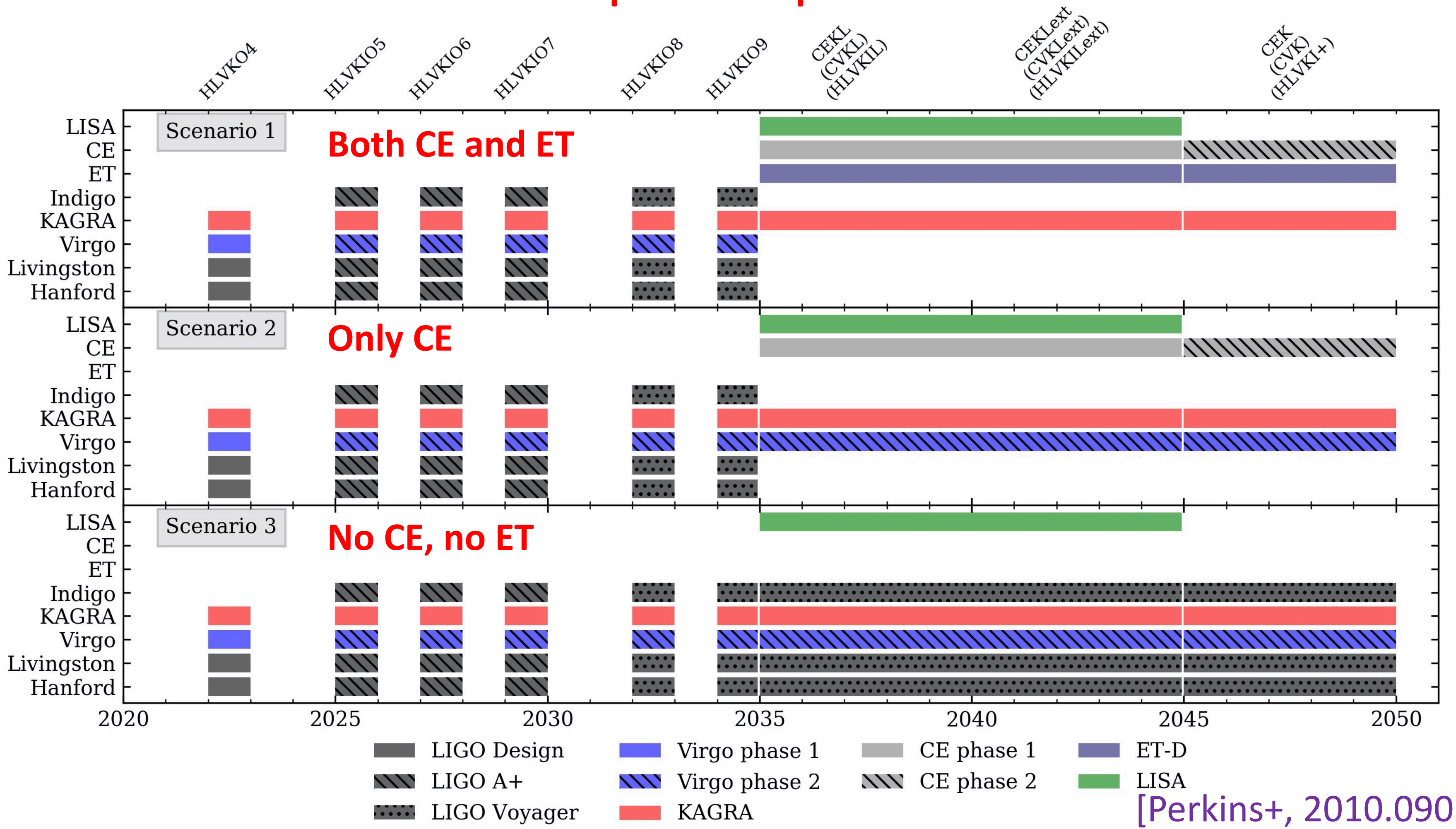
Inspiral: GR solution known, parametrized post-Einstein formalism

$$\tilde{h}(f) = \tilde{A}_{\rm GR}(f)\left[1+\alpha_{\rm ppE}\, v(f)^a\right] e^{i\Psi_{\rm GR}(f)+i\beta_{\rm ppE}\, v(f)^b}$$



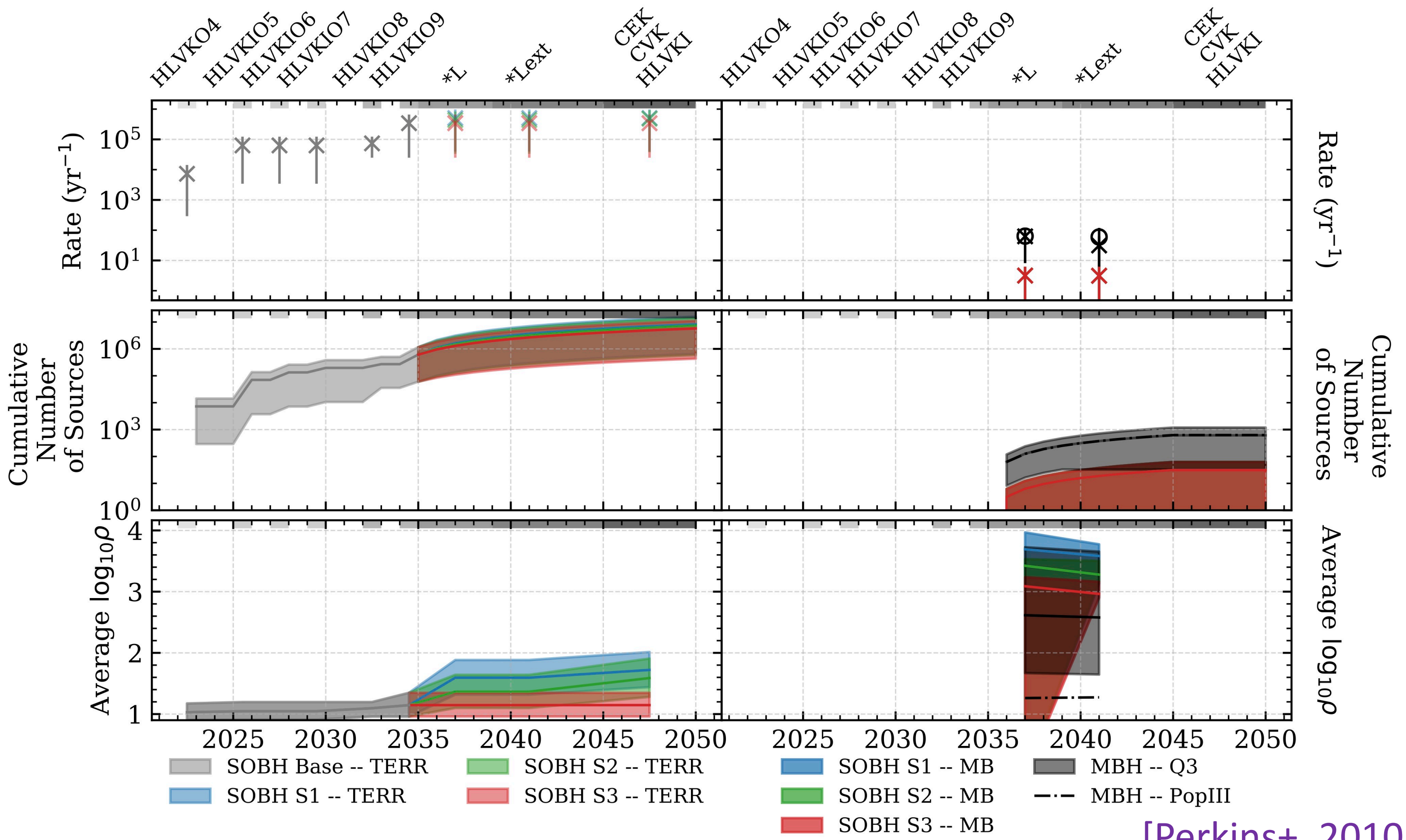
[Yunes-Pretorius+, 0909.3328; Perkins-Yunes-EB, 2010.09010]

How will the bounds improve? Depends on detector timeline



[Perkins+, 2010.09010]

How will the bounds improve? Number of sources and SNR evolution over time



[Perkins+, 2010.09010]

Generic bounds at different PN orders

$$\tilde{h}\left(\vec{\lambda}_{\text{PhenomPv2}}, \beta\right) = \tilde{h}_{\text{GR}} e^{i\beta(\mathcal{M}\pi f)^{b/3}}$$

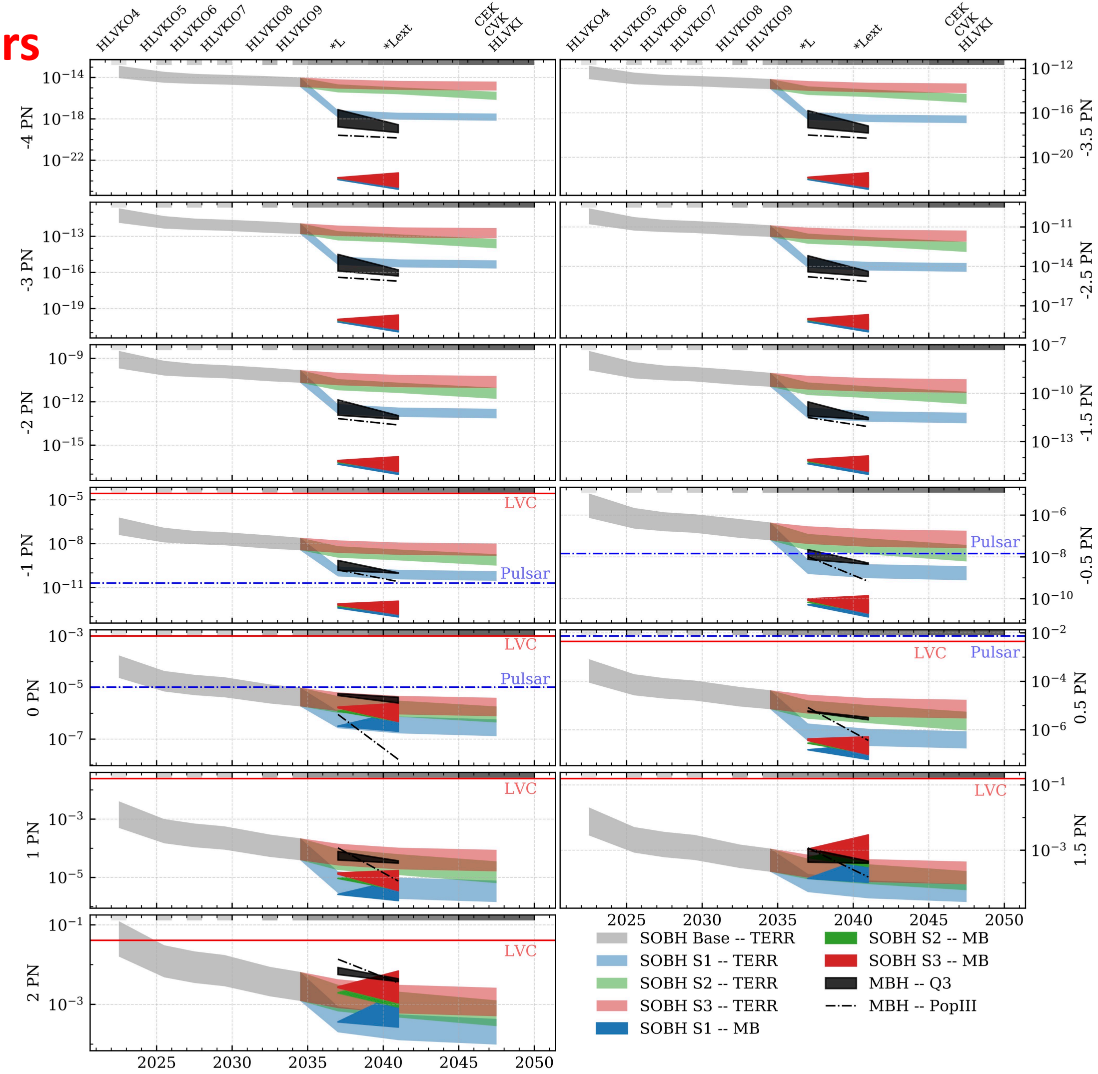
A term $(\pi \mathcal{M} f)^{b/3}$ in the phasing is of $(b+5)/2$ PN order

Multiband (MB) sources best at negative PN orders

MBH better than SOBH at negative PN orders

Terrestrial slightly better than LISA MBHs at positive PN orders

Terrestrial network improvement matter most at negative PN orders

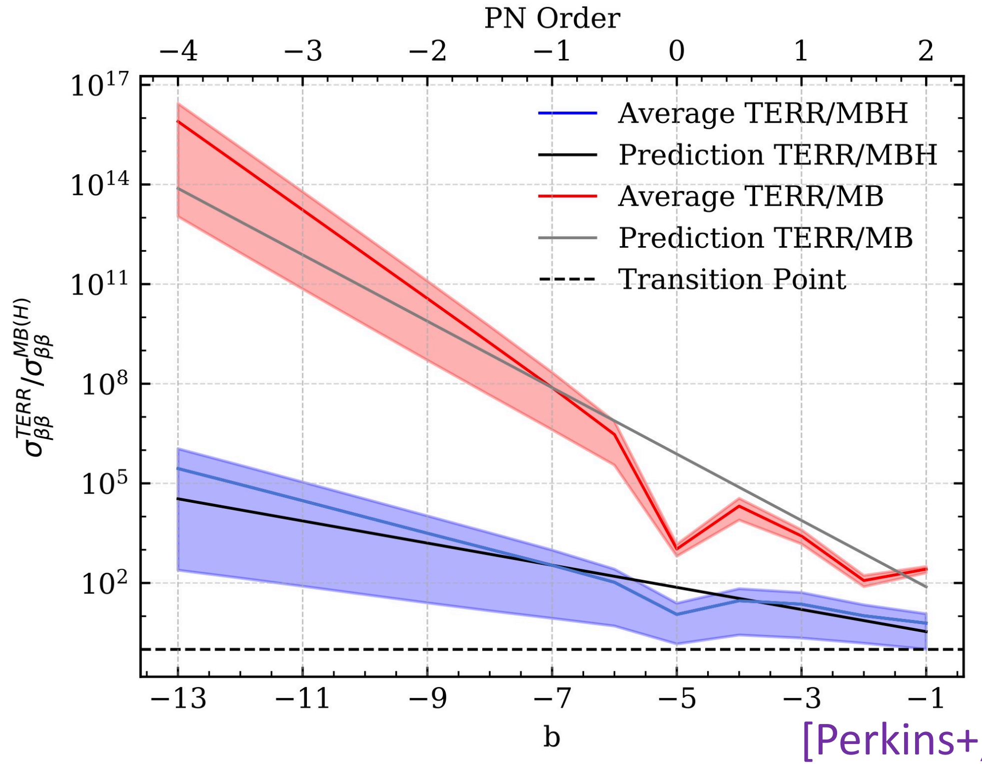


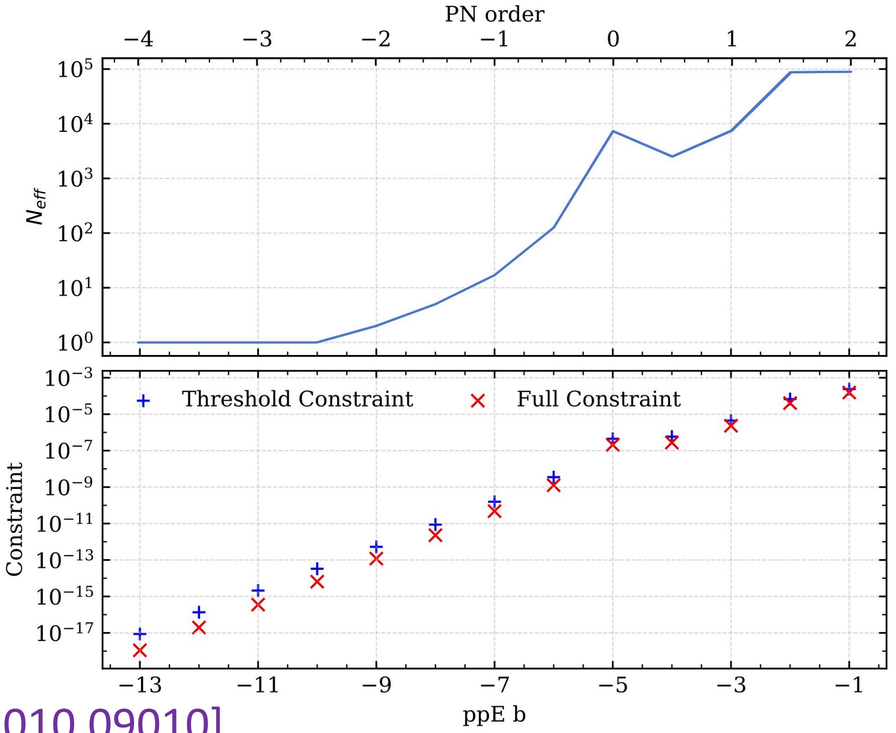
Bounds: are Earth-based sources better? Analytic scaling and N_{eff}

$$\sigma_{\beta\beta} \approx \left[6^{b-2}\left(\frac{b}{2}-1\right)\right]^{1/2} \frac{(\pi \mathcal{M} f_{\rm low})^{-2/3}}{\eta^{(b-2)/5}\rho}, \quad b > 2$$

$$\sigma_{\beta\beta} \approx \left(1-\frac{b}{2}\right)^{1/2} \frac{(\pi \mathcal{M} f_{\rm low})^{-b/3}}{\rho}, \quad b < 2$$

$$\frac{\sigma_{\beta\beta}^{\rm TERR}}{\sigma_{\beta\beta}^{\rm MBH}} \approx \frac{\rho^{\rm MBH}}{\rho^{\rm TERR}} \left(\frac{\mathcal{M}^{\rm TERR}}{\mathcal{M}^{\rm MBH}}\right)^{-b/3} \left(\frac{f_{\rm low}^{\rm TERR}}{f_{\rm low}^{\rm MBH}}\right)^{-b/3}$$





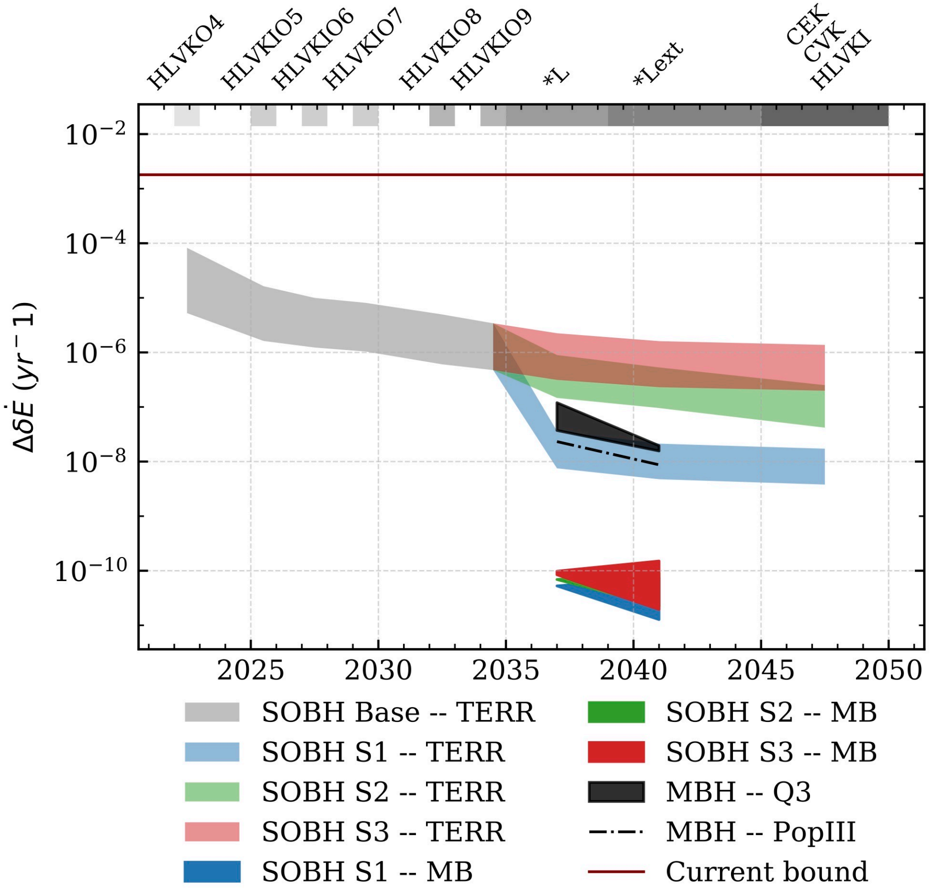
[Perkins+, 2010.09010]

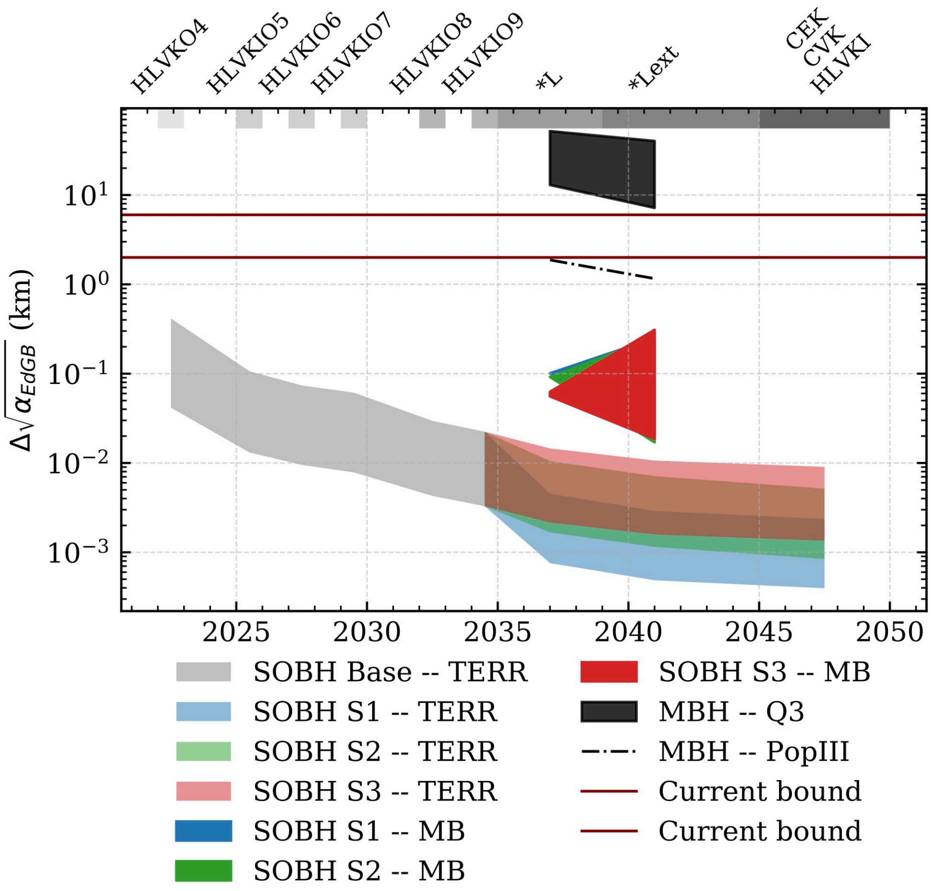
Mapping to specific theories

Table 2 Mapping of ppE parameters to those in each theory for a black hole binary

Theory	β_{ppE}	b
Scalar–tensor [36, 179, 180]	$-\frac{5}{1792}\dot{\phi}^2\eta^{2/5}\left(m_1 s_1^{\text{ST}} - m_2 s_2^{\text{ST}}\right)^2$	-7
EdGB, D^{2}GB [23]	$-\frac{5}{7168}\zeta_{\text{GB}}\frac{\left(m_1^2 s_2^{\text{GB}} - m_2^2 s_1^{\text{GB}}\right)^2}{m^4\eta^{18/5}}$	-7
dCS [181]	$\frac{1549225}{11812864}\frac{\zeta_{\text{CS}}}{\eta^{14/5}}\left[\left(1-\frac{231808}{61969}\eta\right)\chi_s^2 + \left(1-\frac{16068}{61969}\eta\right)\chi_a^2 - 2\delta_m\chi_s\chi_a\right]$	-1
EA [182]	$-\frac{3}{128}\left[\left(1-\frac{c_{14}}{2}\right)\left(\frac{1}{w_2^{\text{Æ}}} + \frac{2c_{14}c_+^2}{(c_+ + c_- - c_- c_+)^2 w_1^{\text{Æ}}} + \frac{3c_{14}}{2w_0^{\text{Æ}}(2-c_{14})}\right) - 1\right]$	-5
Khronometric [182]	$-\frac{3}{128}\left[(1-\beta_{\text{KG}})\left(\frac{1}{w_2^{\text{KG}}}\frac{3\beta_{\text{KG}}}{2w_0^{\text{KG}}(1-\beta_{\text{KG}})}\right) - 1\right]$	-5
Extra dimension [183]	$\frac{25}{851968}\left(\frac{dm}{dt}\right)\frac{3-26\eta+34\eta^2}{\eta^{2/5}(1-2\eta)}$	-13
Varying G [151]	$-\frac{25}{65536}\dot{G}\mathcal{M}$	-13
Mod. disp. rel. [184]	$\frac{\pi^{2-\alpha_{\text{MDR}}}}{(1-\alpha_{\text{MDR}})}\frac{D_{\alpha_{\text{MDR}}}}{\lambda_{\mathbb{A}}^{2-\alpha_{\text{MDR}}}}\frac{\mathcal{M}^{1-\alpha_{\text{MDR}}}}{(1+z)^{1-\alpha_{\text{MDR}}}}$	$3(\alpha_{\text{MDR}} - 1)$

Mapping to theories: two examples (dipolar radiation and EdGB)





Bounds: improvements (generic vs. specific)

PN order (ppE b)	Current Constraint	Best (Worst) Constraint	Best (Worst) Source Class
-4 (-13)	–	10^{-25} (10^{-14})	MB (T)
-3.5 (-12)	–	10^{-23} (10^{-14})	MB (T)
-3 (-11)	–	10^{-21} (10^{-12})	MB (T)
-2.5 (-10)	–	10^{-19} (10^{-11})	MB (T)
-2 (-9)	–	10^{-17} (10^{-10})	MB (T)
-1.5 (-8)	–	10^{-15} (10^{-9})	MB (T)
-1 (-7)	2×10^{-11}	10^{-13} (10^{-11})	MB (MBH)
-0.5 (-6)	1.4×10^{-8}	10^{-11} (10^{-8})	MB (T)
0 (-5)	1.0×10^{-5}	10^{-7} (10^{-5})	MBH (T)
.5 (-4)	$4.4 \times 10^{-3*}$	10^{-7} (10^{-5})	MB (T)
1 (-3)	$2.5 \times 10^{-2*}$	10^{-6} (10^{-4})	MB/T (T)
1.5 (-2)	0.15^*	10^{-5} (10^{-3})	T (MB)
2 (-1)	0.041^*	10^{-4} (10^{-2})	T (MB)

[Perkins+, 2010.09010]

Theory	Parameter	Current bound	Most (Least) Stringent Forecasted Bound	Most (Least) Constraining Class
Generic Dipole	$\delta\dot{E}$	1.1×10^{-3} [44, 45]*	10^{-11} (10^{-6})	MB (T)
Einstein-dilaton-Gauss-Bonnet	$\sqrt{\alpha_{\rm EdGB}}$	1 km [46] 3.4 km [47]*	10^{-3} km (1 km)	T (MBH)
Black Hole Evaporation	$\dot{M}$	–	10^{-8} (10^{2})	MB (T)
Time Varying G	$\dot{G}$	$10^{-13} - 10^{-12}\text{yr}^{-1}$ [48–52]	10^{-9}yr^{-1} (10yr^{-1})	MB (T)
Massive Graviton	m_g	10^{-29}eV [53–56] 10^{-23}eV [3, 57]*	10^{-26} eV $(10^{-24}$ eV)	MBH (MB)
dynamic Chern Simons	$\sqrt{\alpha_{\rm dCS}}$	5.2 km [58]	10^{-2} km (10 km)	T (MB)
Non-commutative Gravity	$\sqrt{\Lambda}$	2.1 [59]*	10^{-3} (10^{-1})	T (MB)

Can we parametrize ringdown in modified gravity theories?

Can we parametrize ringdown? Scalar/EM/gravitational perturbations in GR

Gravitational perturbations of a Schwarzschild BH: Regge-Wheeler/Zerilli equations

$$f\frac{d}{dr}\left(f\frac{d\Phi}{dr}\right)+\left[\omega^2-fV_\pm\right]\Phi=0 \qquad f=1-\frac{r_H}{r}$$

Isospectrality: the odd/even potentials

$$V_-=\frac{\ell(\ell+1)}{r^2}-\frac{3r_H}{r^3}$$

$$V_+=\frac{9\lambda r_H^2 r+3\lambda^2 r_H r^2+\lambda^2(\lambda+2)r^3+9r_H^3}{r^3(\lambda r+3r_H)^2}$$

have the same quasinormal mode spectrum [Chandrasekhar-Detweiler 1975]

Scalar, electromagnetic and (odd) gravitational perturbations:

$$V_s=\frac{\ell(\ell+1)}{r^2}+(1-s^2)\frac{r_H}{r^3}$$

[e.g. EB+, 0905.2975]

Generic (but decoupled) corrections to GR potentials

Modifications to the gravity sector and/or beyond Standard Model physics: expect

- small modifications to the functional form of the potentials – parametrize!
- coupling between the wave equations (more later)

$$V = V_\pm + \delta V_\pm \qquad \delta V_\pm = \frac{1}{r_H^2}\sum_{j=0}^{\infty}\alpha_j^\pm\left(\frac{r_H}{r}\right)^j \qquad \omega_{\mathrm{QNM}}^\pm = \omega_0^\pm + \sum_{j=0}^{\infty}\alpha_j^\pm e_j^\pm$$

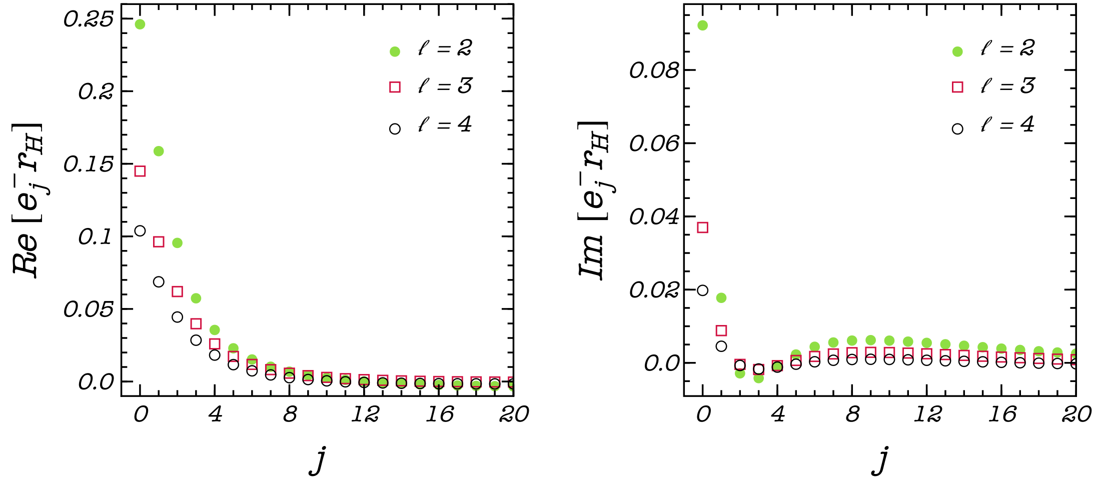
$$V = V_s + \delta V_s \qquad \delta V_s = \frac{1}{r_H^2}\sum_{j=0}^{\infty}\beta_j^s\left(\frac{r_H}{r}\right)^j \qquad \omega_{\mathrm{QNM}}^s = \omega_0^s + \sum_{j=0}^{\infty}\beta_j^s d_j^s$$

Maximum of $f(r)\alpha_j^\pm\left(\frac{r_H}{r}\right)^j$ is $\alpha_j^\pm\frac{(1+1/j)^{-j}}{j+1}$, so corrections are small if:

$$(\alpha_j^\pm,\beta_j^s) \ll (1+1/j)^j(j+1)$$

[Cardoso+, 1901.01265]

Correction coefficients and their asymptotic behavior

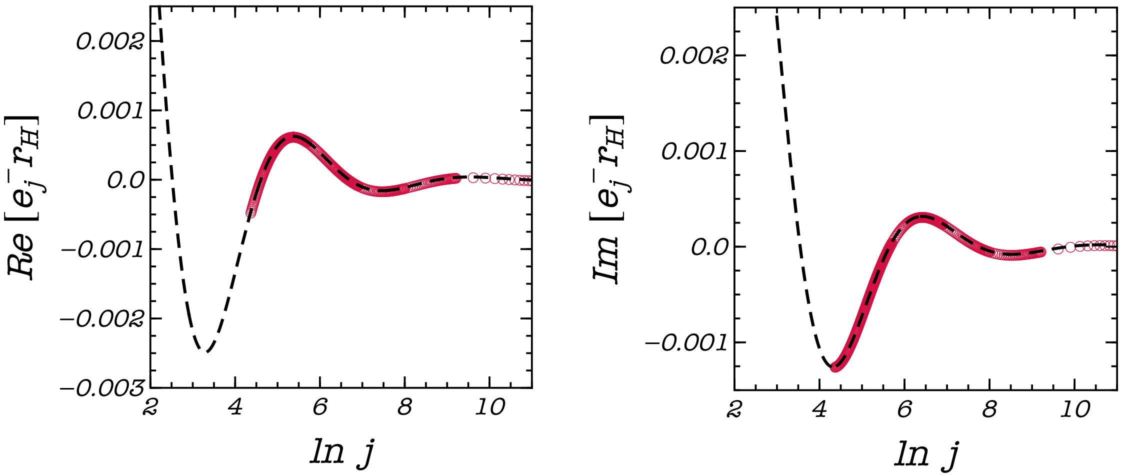


QNM frequency correction coefficients by direct integration [Pani, 1305.6759]

Asymptotics:

$$e_j \propto \frac{j^{2i\omega_0 r_H}}{j} = \frac{e^{2ir_H\omega_R \ln j}}{j^{1+2r_H\omega_I}}$$

Damped oscillatory behavior for large j



Fitting the numerics by

$$e_j \sim \frac{\kappa}{j^\beta}\sin(\gamma \ln j + \zeta)$$

confirms this.

[Pani, 1305.6759]

Generic isospectrality breaking

Isospectrality follows from the existence of a "superpotential" such that:

$$fV_{\pm} = W_0^2 \mp f\frac{dW_0}{dr} - \frac{\lambda^2(\lambda+2)^2}{36r_H^2} \qquad W_0 = \frac{3r_H(r_H - r)}{r^2(3r_H + \lambda r)} - \frac{\lambda(\lambda+2)}{6r_H}$$

Perturb to find conditions for isospectrality to hold:

$$2\frac{d\delta W}{dr} = \delta V_- - \delta V_+ \qquad 4\frac{W_0}{f}\delta W = \delta V_+ + \delta V_-$$

Preserving isospectrality needs fine tuning!

$$\alpha_0^+ = \alpha_0^-$$
$$\alpha_1^+ = \alpha_1^-$$
$$\alpha_2^+ = \alpha_2^- + \frac{6\left(\alpha_0^- - \alpha_1^-\right)}{\lambda(\lambda+2)}$$

[Chandrasekhar-Detweiler 1975]

Example 1: EFT

EFT corrections quartic in the curvature lead to a modified Regge-Wheeler equation:

$$\frac{d^2\Psi_-}{dr_\star^2} + \left[\omega^2 - f(V_- + \delta V_-)\right]\Psi_- = 0$$

$$\delta V_- = \epsilon_2 \frac{18(\ell+2)(\ell+1)(\ell-1)r_H^8}{r^{10}}$$

Trivially read off the correction coefficient: $\alpha_{10}^{-} = 18(\ell+2)(\ell+1)(\ell-1)\epsilon_2$

Plug into $\omega_{\rm QNM}^{\pm} = \omega_0^{\pm} + \sum_{j=0}^{\infty} \alpha_j^{\pm} e_j^{\pm}$

to find

$$r_H\omega = r_H\omega_0 + (0.0663354 + 0.117439\mathrm{i})\epsilon_2(\ell-1)(\ell+1)(\ell+2)$$

in agreement with numerical integrations.

[Cardoso+, 1808.08962]

Example 2: Reissner-Nordström

Odd gravitational perturbations of Reissner-Nordström satisfy

$$f_{\mathrm{RN}}\frac{d}{dr}\left(f_{\mathrm{RN}}\frac{d\Phi}{dr}\right)+\left(\omega^2-f_{\mathrm{RN}}V_{\mathrm{RN}}\right)\Phi=0 \qquad f_{\mathrm{RN}}=1-\frac{2M}{r}+\frac{Q^2}{r^2}$$

A simple change of variables brings the wave equation in our "canonical" form, with

$$V_{\mathrm{RN}}=\frac{\ell(\ell+1)}{r^2}+\frac{4r_Hr_-}{r^4}-\frac{3(r_H+r_-)}{2r^3}-\frac{\left[4(\ell-1)(\ell+2)r_Hr_-+\frac{9}{4}(r_H+r_-)^2\right]^{1/2}}{r^3}$$

for small charge.

Read off coefficients to find:

$$\omega_{\mathrm{QNM}}=\left(1-\frac{r_-}{r_H}\right)\left(\frac{2\Omega_0}{r_H}+e_0\alpha_0^-+e_3\alpha_3^-+e_4\alpha_4^-\right)$$

$$=\frac{\Omega_0}{M}+\frac{(0.0258177-0.002824i)Q^2}{M^3}$$

TABLE II. Relative percentage errors on the real and imaginary parts of the QNMs for RN BHs, as a function of the charge-to-mass ratio Q/M.

Q/M	Δ_R	Δ_I
0.00	0%	0%
0.05	0.11%	0.042%
0.10	0.43%	0.17%
0.20	1.7%	0.66%
0.30	3.8%	1.5%
0.40	6.8%	2.6%
0.50	11%	4.2%

Example 3: Klein-Gordon in slowly rotating Kerr

At linear order in the spin parameter:

$$f\frac{d}{dr}\left(f\frac{d}{dr}\right)\Phi + \left(\omega^2 - fV_0 - \frac{4amM\omega}{r^3}\right)\Phi = 0$$

i.e.

$$f\frac{d}{dr}\left(f\frac{d}{dr}\right)\Phi + \left[\left(\omega - \frac{am}{r_H^2}\right)^2 - f\left(V_0 - \frac{2am\omega}{r_H^2} - \frac{2am\omega}{r_H^2}\frac{r_H}{r} - \frac{2am\omega}{r_H^2}\left(\frac{r_H}{r}\right)^2\right)\right]\Phi = 0$$

Correction coefficients to the scalar wave equation:

$$\beta_0^0 = \beta_1^0 = \beta_2^0 = -2am\omega_0^0$$

$$\omega_{\rm QNM} = \omega_0^0 + \frac{am}{r_H^2} - 2am\omega_0^0(d_0^0 + d_1^0 + d_2^0)$$

TABLE III. Relative percentage errors in the real and imaginary parts of the QNM frequencies for scalar perturbations around a slowly spinning black hole, as a function of the BH angular momentum a/M.

a/M	Δ_R	Δ_I
0	0%	0%
10^{-4}	0.0050%	0.83%
10^{-3}	0.049%	5.1%
10^{-2}	0.49%	34%

Coupled perturbations

We really want to solve the coupled $N \times N$ system

$$f\frac{d}{dr}\left(f\frac{d\mathbf{\Phi}}{dr}\right) + \left[\omega^2 - f\mathbf{V}\right]\mathbf{\Phi} = 0 \qquad \mathbf{\Phi} = \{\Phi_i\}\ (i = 1, \ldots, N)$$

$$\mathbf{V}(r) = V_{ij}(r)$$

where each matrix element is perturbed:

$$V_{ij} = V_{ij}^{\mathrm{GR}} + \delta V_{ij} \qquad \delta V_{ij} = \frac{1}{r_H^2}\sum_{k=0}^{\infty}\alpha_{ij}^{(k)}\left(\frac{r_H}{r}\right)^k$$

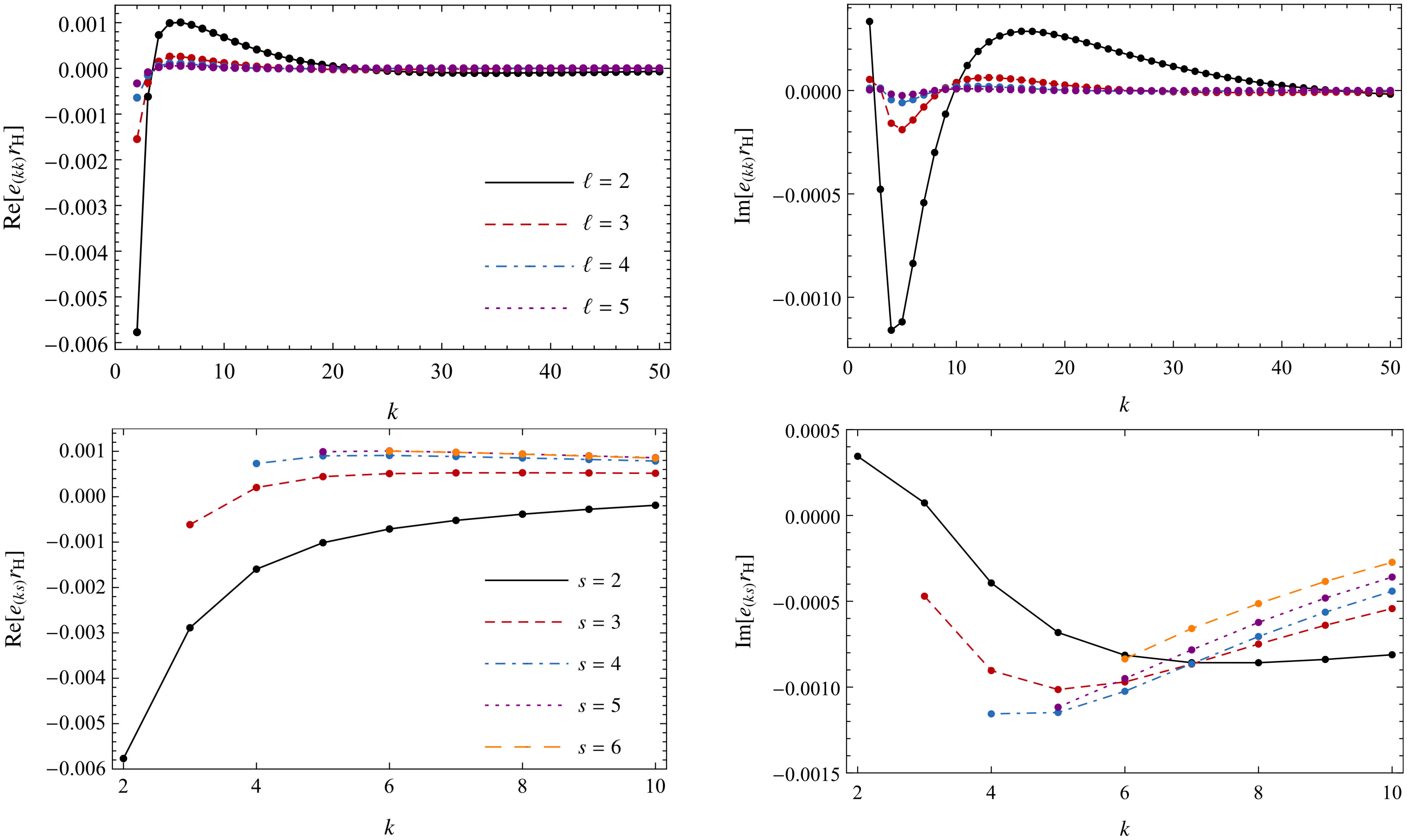
If the background spectra are nondegenerate, coupling will induce **quadratic** corrections. Allow α to depend on ω. We need

- quadratic corrections in α, besides the linear diagonal terms $d_{(k)}^{ii}$
- coupling-induced corrections

$$\omega \approx \omega_0 + \alpha_{ij}^{(k)} d_{(k)}^{ij} + \alpha_{ij}^{(k)} \alpha_{pq}'^{(s)} d_{(k)}^{ij} d_{(s)}^{pq} + \frac{1}{2}\alpha_{ij}^{(k)}\alpha_{pq}^{(s)} e_{(ks)}^{ijpq} \qquad \text{(Einstein summation)}$$

[McManus+, 1906.05155]

Correction coefficients



The degenerate case

Degenerate spectra (e.g. **even/odd gravitational perturbations**) need special care. Why?

$$\left(\frac{d^2}{dr_*^2}+\omega^2-fV_0\right)\phi_1+\alpha Z\phi_2=0$$
$$\left(\frac{d^2}{dr_*^2}+\omega^2-fV_0\right)\phi_2+\alpha Z\phi_1=0$$

Diagonalize:

$$\phi_1=(\phi_++\phi_-)/2$$
$$\phi_2=(\phi_+-\phi_-)/2$$

$$\left(\frac{d^2}{dr_*^2}+\omega^2-fV_0+\alpha Z\right)\phi_+=0$$
$$\left(\frac{d^2}{dr_*^2}+\omega^2-fV_0-\alpha Z\right)\phi_-=0$$

Corrections are linear in α

Use degenerate perturbation theory:

$$\omega=\omega_0+\epsilon\omega_1 \qquad \omega_1=\frac{\delta V_{++}+\delta V_{--}\pm\sqrt{(\delta V_{++}-\delta V_{--})^2+4\delta V_{+-}\delta V_{-+}}}{2}$$

$$\delta V_{\pm\pm}=\sum_{k=0}^{\infty}\alpha_{\pm\pm}^{(k)}\left\langle\omega_0,\pm\middle|f(r)\frac{r_H^{k-2}}{r^k}\middle|\omega_0,\pm\right\rangle=\sum_{k=0}^{\infty}\alpha_{\pm\pm}^{(k)}\delta V_{\pm\pm}^{(k)}$$

Example 1: scalar/odd gravitational in dynamical Chern-Simons

Spectra are nondegenerate

The perturbed potentials read:

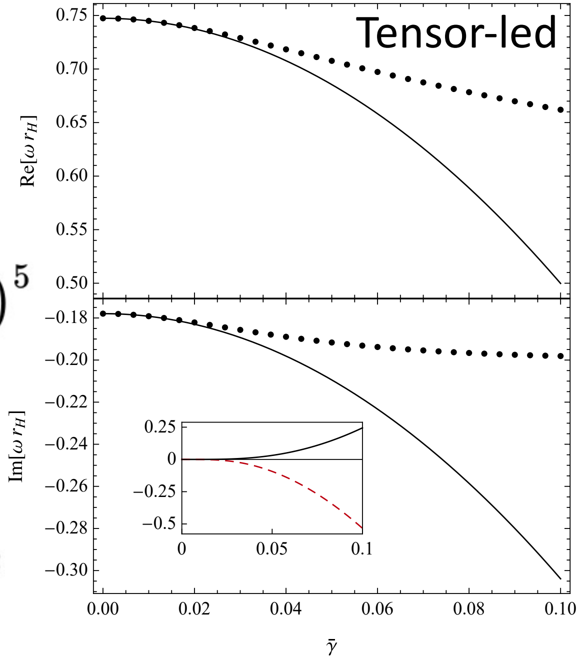
$$V_{11} = V_-$$

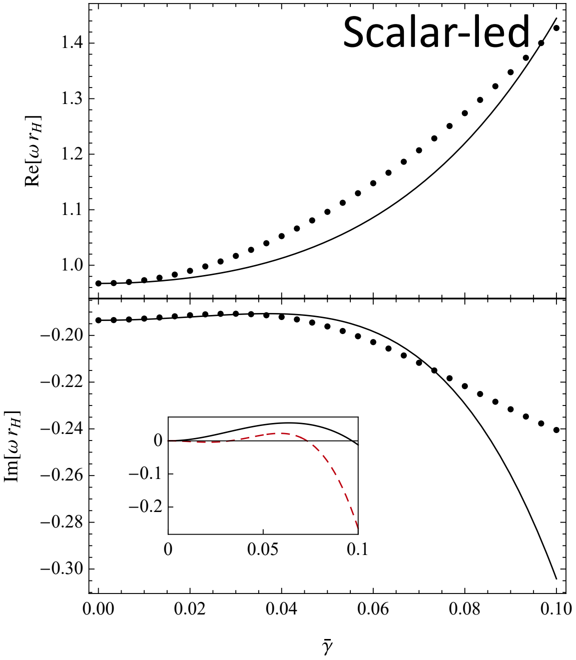
$$V_{12} = V_{21} = \frac{1}{r_H^2}\frac{12}{\sqrt{\beta}r_H^2}\sqrt{\pi\frac{(\ell+2)!}{(\ell-2)!}}\left(\frac{r_H}{r}\right)^5$$

$$V_{22} = V_{s=0} + \frac{1}{r_H^2}\frac{144\pi\ell(\ell+1)}{\beta r_H^4}\left(\frac{r_H}{r}\right)^8$$

Corrected frequencies:

$$\omega = \omega_0 + e^{1221}_{(55)}\left(12\bar{\gamma}\sqrt{\pi\frac{(\ell+2)!}{(\ell-2)!}}\right)^2$$





$$\omega = \omega_0 + 2d_{(8)}144\pi\ell(\ell+1)\bar{\gamma}^2 + e_{(88)}\left[144\pi\ell(\ell+1)\bar{\gamma}^2\right]^2 + e^{1221}_{(55)}\left(12\bar{\gamma}\sqrt{\pi\frac{(\ell+2)!}{(\ell-2)!}}\right)^2$$

[Cardoso-Gualtieri, 0907.5008; Molina+, 1004.4007]

Example 2: scalar-led perturbations in Horndeski

The scalar-led perturbation is related to background coupling functions in the Horndeski Lagrangian:

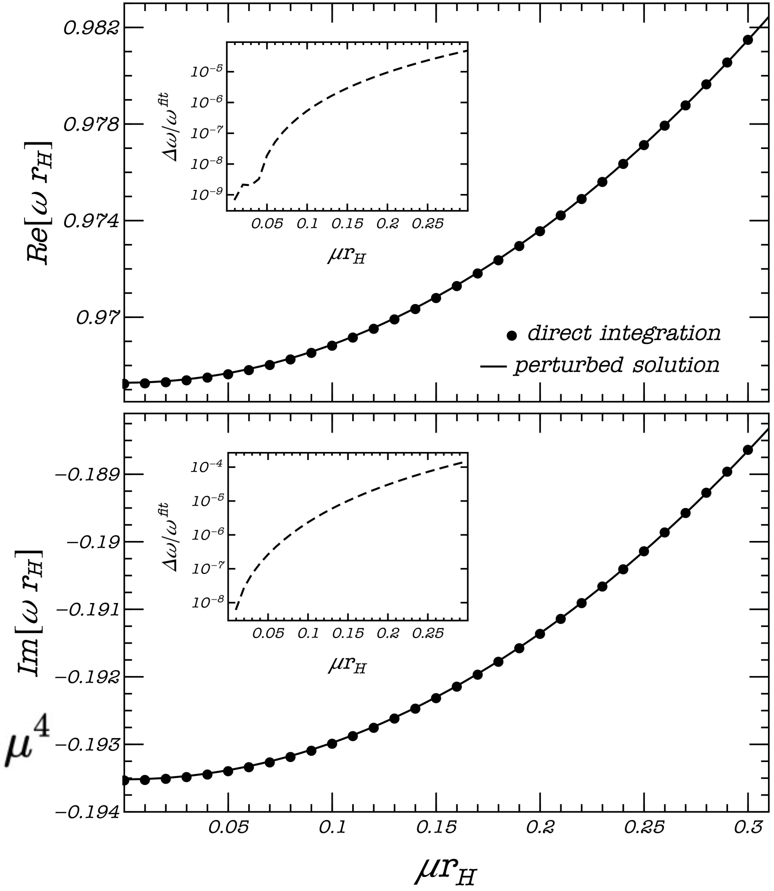
$$\frac{d^2\phi}{dr_*^2} + \left[\omega^2 - f\left(V_{s=0} + \mu^2 + \frac{\ell(\ell+1)}{r^2} f\Gamma\right)\right]\phi = 0$$

$$\mu^2 = \frac{-\overline{G}_{2\phi\phi}}{3\overline{G}_{4\phi}^2 + \overline{G}_{2X} - 2\overline{G}_{3\phi}}$$

$$\Gamma = \frac{8\overline{G}_{4X}}{3\overline{G}_{4\phi}^2 + \overline{G}_{2X} - 2\overline{G}_{3\phi}}$$

Corrected frequencies read (can set $\Gamma = 0$):

$$\omega \approx \omega_0 + d_{(0)}\mu^2 + \left[d_{(2)} + r_H d_{(3)}\right]\ell(\ell+1)\Gamma + \frac{1}{2}e_{(00)}\mu^4 + \frac{1}{2}\left[e_{(22)} + 2r_H e_{(23)} + r_H^2 e_{(33)}\right]\left[\ell(\ell+1)\Gamma\right]^2 + \left[e_{(02)} + r_H e_{(03)}\right]\ell(\ell+1)\mu^2\Gamma$$



[Tattersall+, 1711.01992]

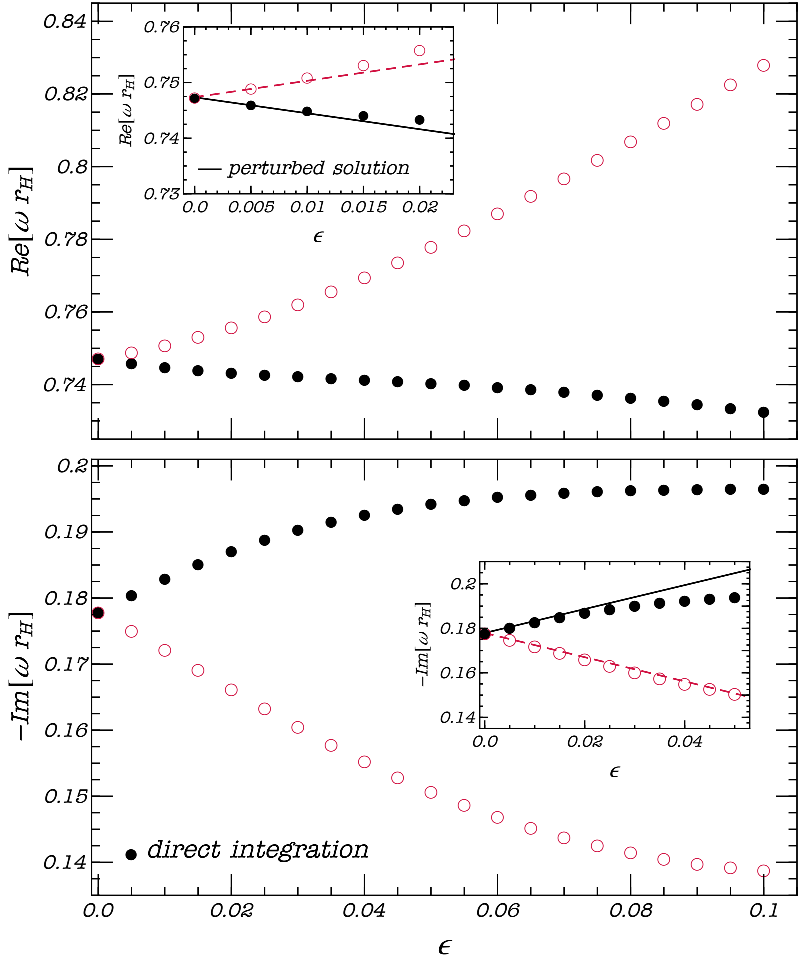
Example 3: odd/even gravitational coupling in EFT (degenerate)

The quartic-in-curvature EFT leads to a degenerate perturbed eigenvalue problem:

$$V_{11} = V_+$$
$$V_{22} = V_-$$
$$V_{12} = V_{21} = \epsilon V(r)$$

where off-diagonal perturbations are given in [Cardoso+, 1808.08962]

Direct integration vs. degenerate parametrization: good agreement, but quadratic corrections could be useful



Parametrized merger/ringdown: a summary

Modifications to the gravity sector and/or beyond Standard Model physics:

- small modifications to the potentials
- coupling between the (matrix-valued) wave equations

We parametrized modifications by power laws, then computed perturbed QNMs for:

- linear corrections to diagonal terms [Cardoso+, 1901.01265]
- quadratic corrections + coupling [McManus+, 1906.05155]

General formalism – unless you can't find wave equations [Langlois+ 2103.14750]
Examples:

- EFT, Reissner-Nordström, Klein-Gordon in Kerr for slow rotation
- scalar/odd gravitational dCS, scalar-led Horndeski, odd/even gravitational EFT

Needed generalizations:

- higher-order corrections (in particular, in degenerate coupled case)
- **coupled gravitational modes with rotation – LIGO/Virgo remnants have spins 0.7 or so!**

Rotating BH QNMs in modified gravity: the EFT viewpoint

QNM calculations: limited sample (EdGB/EsGB, dCS), mostly **nonrotating** BHs
[Blazquez-Salcedo+ 1609.01286 (EdGB), 2006.06006 (EsGB); Molina+ 1004.4007 (dCS)]
Cano's work: systematic small-rotation expansion + **scalar** QNMs
Theories: sum over curvature invariants with scalar-dependent coefficients

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\left[R + \sum_{n=2}^{\infty}\ell^{2n-2}\mathcal{L}_{(n)}\right]$$

and more specifically, at order ℓ^4

EsGB dCS (dilaton+axion)

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{|g|}\Big\{R + \alpha_1\phi_1\ell^2 R_{\rm GB} + \alpha_2(\phi_2\cos\theta_m + \phi_1\sin\theta_m)\ell^2 R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$$
$$+\lambda_{\rm ev}\ell^4 R_{\mu\nu}^{\rho\sigma}R_{\rho\sigma}^{\delta\gamma}R_{\delta\gamma}^{\mu\nu} + \lambda_{\rm odd}\ell^4 R_{\mu\nu}^{\rho\sigma}R_{\rho\sigma}^{\delta\gamma}\tilde{R}_{\delta\gamma}^{\mu\nu} - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2\Big\}$$

Einsteinian cubic gravity (+parity-breaking) - causality constraints [Camanho+ 1407.5597]
Next order, no new DOFs [Endlich-Gorbenko-Huang-Senatore, 1704.01590]

$$S_{(4)} = \frac{\ell^6}{16\pi G}\int d^4x\sqrt{|g|}\left\{\epsilon_1\mathcal{C}^2 + \epsilon_2\tilde{\mathcal{C}}^2 + \epsilon_3\mathcal{C}\tilde{\mathcal{C}}\right\}$$

$$\mathcal{C} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \quad \tilde{\mathcal{C}} = R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$$

[Cano-Ruipérez, 1901.01315; Cano-Fransen-Hertog, 2005.03671. See also work by Hui, Penco…]

Calculations of rotating BH QNMs in modified gravity: the EFT viewpoint

Background solutions:
algorithm to compute small-coupling corrections, up to order 14 in rotation

Scalar QNM calculations: "quasi-separable"
For zero coupling, can be separated in terms of spin-weighted **spheroidal** harmonics

$$\nabla^2\psi = \int_{-\infty}^{\infty} d\omega \sum_{m=-\infty}^{\infty} e^{i(m\phi-\omega t)}\mathcal{D}^2_{m,\omega}\psi_{m,\omega}$$

$$\mathcal{D}^2_{m,\omega} = \mathcal{D}^2_{(0)m,\omega} + \lambda\mathcal{D}^2_{(1)m,\omega}$$

$$\psi_{m,\omega} = \sum_{l=|m|}^{\infty} S_{l,m}(x;a\omega)R_{l,m}(\rho)$$

In summary: second-order radial ODEs can be cast as wave equations via redefinitions of the radial variable/radial WF, and solved either numerically or via WKB

$$\frac{d^2\varphi}{dy^2} + \left(\omega^2 - V(y;\omega)\right)\varphi = 0$$

Note: not all potentials vanish at the horizon

[Cano+ 1901.01315, 2005.03671; for EsGB, see also Pierini-Gualtieri, 2103.09870]

What do we learn from these parametrizations?

Parametrized spectroscopy: how many observations do we need?

Use a small-spin expansion and add parametric deviations to frequency and damping time
Assume you detect **N** sources, and **q** QNM frequencies for each source

Order in the spin expansion: need at least 4 or 5 in GR

$J = 1, 2, \ldots, q$ modes/source

$$\omega_i^{(J)} = \frac{1}{M_i} \sum_{n=0}^{D} \chi_i^n w_J^{(n)} \left(1 + \gamma_i \delta w_J^{(n)}\right)$$

$i = 1, \ldots, N$ sources

$$\tau_i^{(J)} = M_i \sum_{n=0}^{D} \chi_i^n t_J^{(n)} \left(1 + \gamma_i \delta t_J^{(n)}\right)$$

Expansion coefficients in GR

Small, universal non-GR corrections

How many parameters?
If $\gamma_i = \alpha$ for all sources ,
reabsorb $\gamma_i \delta w^{(n)} \to \delta w^{(n)}$

$\mathcal{P} = 2(D+1)q \longrightarrow D = 4$

- $q = 1$, $\ell = m = 2$ $\longrightarrow \mathcal{P} = 10$
- $q = 2$, $\ell = m = 2$, $\ell = m = 3$ $\longrightarrow \mathcal{P} = 20$

How many observables? $\mathcal{O} = 2N \times q$

Need only $N \geq D + 1$

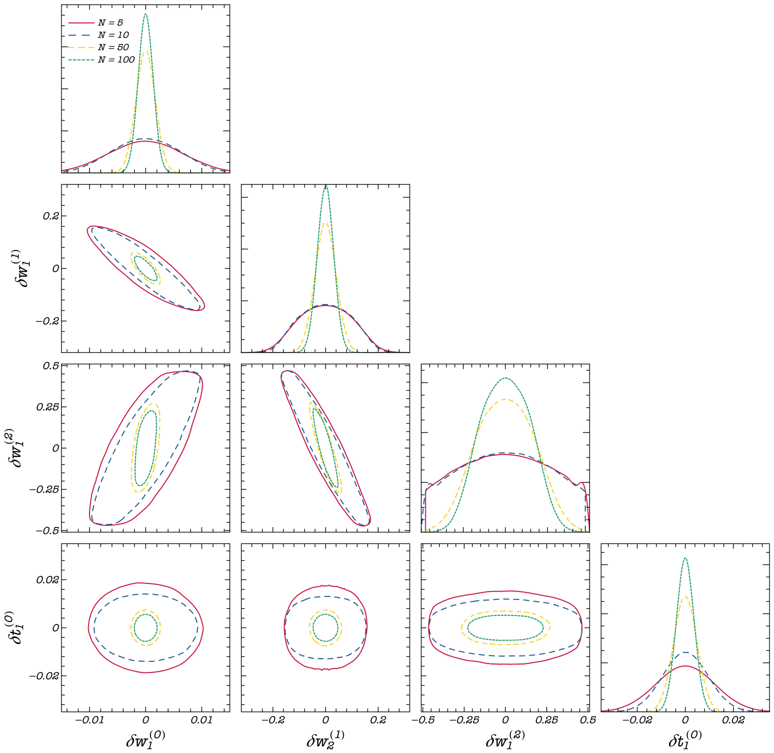
[Maselli+, 1711.01992]

Parametrized spectroscopy: a proof of principle

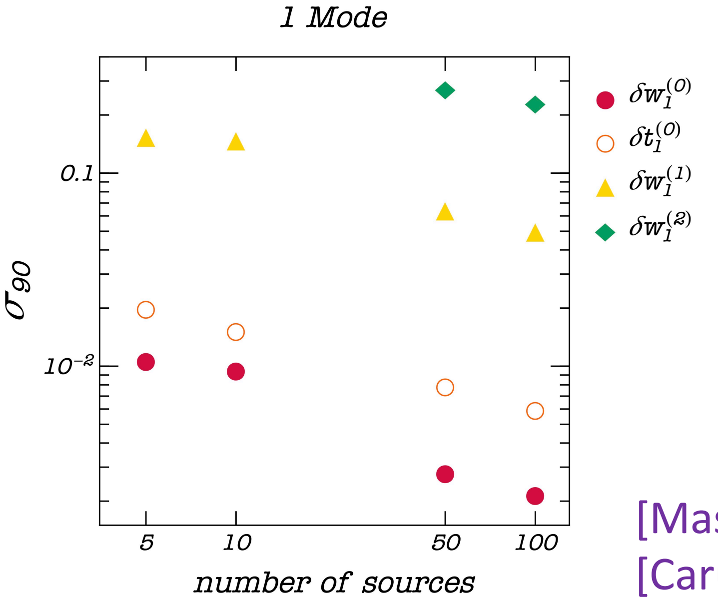
Complication: the coupling is often dimensionful $\gamma_i = \frac{\alpha}{(M_i^{\rm s})^p} = \frac{\alpha(1+z_i)^p}{M_i^p}$

Use Bayesian inference (MCMC), $p = 0$, $q = 1$ (one mode), simple source distributions

Einstein Telescope: constrain first three frequency coeffs and only the first damping coeffs



Width at 90% confidence gets better as we get more observations:



[Maselli+, 1711.01992]
[Carullo, 2102.05939]

Take-home messages

Black hole solutions beyond GR:
Stringent no-hair theorems: Kerr solution is still a solution in most beyond-GR theories
There are loopholes (e.g., EsGB)
Dipolar radiation from black hole binaries (e.g. in EsGB) can be tested in the inspiral (ppE)
Curvature/spin induced scalarization can be tested in inspiral, merger and ringdown

What can we say about beyond-GR black holes with gravitational waves?
ppE, black hole spectroscopy

Parametrized tests of GR with black hole ringdown:
Nonrotating case quite well understood – but irrelevant to most "real" mergers
No parametrization if equations can't be case in (coupled) Schrödinger-like form

Inverse problem: parametrize deviations; if measured, find the "true" theory of gravity
Technical obstacle: rotation in beyond-GR gravity is hard!