

Smarr formulas and rod structure

Dmitry Galtsov

Moscow State University

Joint work with

Gerard Clement (LAPTH, CNRS)

and **Igor Bogush** (MSU).

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Christodoulou, Hawking, Smarr

Reversible and Irreversible Transformations in Black-Hole Physics*

Demetrios Christodoulou

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

(Received 17 September 1970)

The concepts of irreducible mass and of reversible and irreversible transformations in black holes are introduced, leading to the formula $E^2 = m_{\text{ir}}^2 + (L^2/4m_{\text{ir}}^2) + p^2$ for a black hole of linear momentum p and angular momentum L .

Gravitational Radiation from Colliding Black Holes

S. W. Hawking

Institute of Theoretical Astronomy, University of Cambridge, Cambridge, England

(Received 11 March 1971)

It is shown that there is an upper bound to the energy of the gravitational radiation emitted when one collapsed object captures another. In the case of two objects with equal masses m and zero intrinsic angular momenta, this upper bound is $(2-\sqrt{2})m$.

Mass Formula for Kerr Black Holes

Larry Smarr*

Center for Relativity Theory, Physics Department, The University of Texas at Austin, Austin, Texas 78712

(Received 29 September 1972)

A new mass formula for Kerr black holes is deduced, and is contrasted to the mass formula which is obtained by integrating term by term the mass differential and which consists of three terms interpreted, respectively, as the surface energy, rotational energy, and electromagnetic energy of the charged rotating black hole. A comparison is suggested between a rotating black hole and a rotating liquid drop which leads to a speculation that Kerr black holes may develop instabilities.

$$A = 4\pi[2M^2 + 2(M^4 - L^2 - M^2Q^2)^{1/2} - Q^2] \longrightarrow dM = T dA + \Omega dL + \Phi dQ$$

$$\longrightarrow M = 2TA + 2\Omega L + \Phi Q$$

we say T = effective surface tension, Ω = angular velocity, and Φ = electromagnetic potential.

⁵J. Bekenstein, Ph.D. thesis, Princeton University, 1972 (unpublished).

Carter derivation of Smarr formula

General derivation of Smarr formula for regular stationary axisymmetric Einstein-Maxwell black holes was given by Carter (1973) based on Komar conserved charges

$$M = \frac{1}{4\pi} \oint_{\infty} D^{\nu} k^{\mu} d\Sigma_{\mu\nu},$$

$$J = -\frac{1}{8\pi} \oint_{\infty} D^{\nu} m^{\mu} d\Sigma_{\mu\nu}$$

where $k^{\mu} = \delta_t^{\mu}$ and $m^{\mu} = \delta_{\varphi}^{\mu}$ are the Killing vectors associated with time translations and rotations around the z -axis.

Because the integrand $D^{\nu} k^{\mu}$ is antisymmetric, one can apply the Ostrogradsky theorem to transform

$$M = \sum_n \frac{1}{4\pi} \oint_{\Sigma_n} D^{\nu} k^{\mu} d\Sigma_{\mu\nu} + \frac{1}{4\pi} \int D_{\nu} D^{\nu} k^{\mu} dS_{\mu}, \quad k = \partial/\partial t$$

where Σ_n are the spacelike surfaces bounding the various sources, and the second integral is over the bulk. Using again the fact that k is a Killing vector and the Einstein equations, we obtain

$$D_{\nu} D^{\nu} k^{\mu} = -[D_{\nu}, D^{\mu}] k^{\nu} = -R^{\mu}_{\nu} k^{\nu} = -8\pi T^{\mu}_{\nu} k^{\nu} \quad T^{\mu}_{\nu} = \frac{1}{4\pi} \left[F^{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta^{\mu}_{\nu} F^{\rho\sigma} F_{\rho\sigma} \right]$$

Similarly, for azimuthal Killing vector m , one finds the angular momentum

$$J = - \sum_n \frac{1}{8\pi} \oint_{\Sigma_n} D^\nu m^\mu d\Sigma_{\mu\nu} + \int T^\mu{}_\nu m^\nu dS_\mu, \quad m = \frac{\partial}{\partial\phi}$$

Combining M and J together, one derives Smarr formula for electrovacuum

$$M = \frac{\kappa A}{4\pi} + 2\Omega_H J + \Phi_H Q \quad \text{where} \quad \kappa^2 = - \frac{1}{2} (D^\mu \xi^\nu) (D_\mu \xi_\nu) \Big|_{\mathcal{N}} \quad \text{surface gravity}$$

for rotating Killing vector $\xi = k + \Omega_H m$ which is timelike outside the horizon and becoming null on the horizon

Its differential form gives the first law

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ$$

Note, that the norm of spacelike rotational Killing m becomes null on the polar axis, for Kerr, e.g.,

$$m^2 = g_{\phi\phi} = a^2 \sin^2 \theta \left(1 + \frac{r^2}{a^2} \right) + \frac{Ma^2}{r} \left(\frac{2 \sin^4 \theta}{1 + \frac{a^2}{r^2} \cos^2 \theta} \right)$$

i.e. polar axis is a Killing horizon of m

Taub-NUT

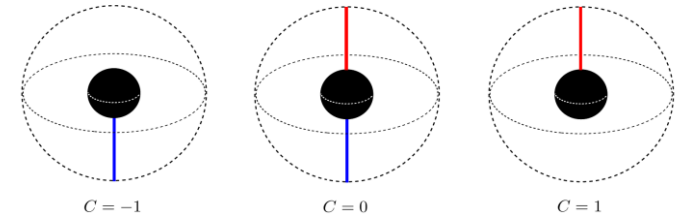
$$ds^2 = -f(r)^2(dt - 2n(\cos\theta + C)d\varphi)^2 + \frac{dr^2}{f(r)^2} + (n^2 + r^2)d\Omega^2$$

$$f(r)^2 = \frac{r^2 - 2GMr - n^2}{n^2 + r^2}$$

Time identification $t \rightarrow t + 1/4n$
makes Misner string non-observable.

Here it is not assumed

Misner string



Role of parameter C

- No curvature singularity at $r=0$, only distributional singularity on the polar axis (Misner string)
- Not asymptotically flat (only locally AF)
- Superposition of NUT and anti NUT gives AF solution with rotation (asymptotically Kerr)
- **Misner string contributes to Smarr-type relation** (Hawking, Hunter, Mann....)

Areas of the back surfaces of an infinitely thin rods are finite per unit length:

$$\lim_{\rho \rightarrow 0} \oint_{\Sigma_\rho} \sqrt{g_{zz}g_{\varphi\varphi}} dz d\varphi \neq 0, \quad \text{finite area}$$

In particular, the area of the horizon is the area of the infinitely thin horizon rod.

Rehabilitating space-times with NUTs

We revisit the Taub–NUT solution of the Einstein equations without time periodicity condition, showing that the Misner string is still fully transparent for geodesics. In this case, analytic continuation can be carried out through both horizons leading to a Hausdorff spacetime without a central singularity, and thus geodesically complete. Furthermore, we show that, in spite of the presence of a region containing closed time-like curves, there are no closed causal geodesics. Thus, some longstanding obstructions to accept the Taub–NUT solution as physically relevant are removed.

NUT wormholes

Phys.Rev. D93 (2016) 024048

We show that supercritically charged black holes with a Newman–Unti–Tamburino (NUT) parameter provide a new setting for traversable wormholes. This does not require exotic matter, but there is a price—the Misner string singularities. Without assuming time periodicity to make Misner strings unobservable, we show that, contrary to expectations, geodesics do not stop there. Moreover, since there is no central singularity, the spacetime turns out to be geodesically complete. Another unpleasant feature of spacetimes with NUTs is the presence of regions where the azimuthal angle ϕ becomes timelike, signalling the appearance of closed timelike curves (CTCs). We show that among them there are no closed timelike or null geodesics, so the freely falling observers should not encounter causality violations.

Weyl coordinates

$$ds^2 = -e^{2U} dt^2 + e^{-2U} [e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\phi^2]$$

$$U_{,\rho\rho} + \frac{1}{\rho} U_{,\rho} + U_{,zz} = 0$$

This may be recognised as Laplace's equation $\nabla^2 U = 0$ for an axially symmetric function in an unphysical Euclidean 3-space in cylindrical polar coordinates, though the coordinates ρ, z, ϕ here have a different meaning.

Once U is known, one has to integrate a system $\gamma_{,\rho} = \rho (U_{,\rho}^2 + U_{,z}^2), \quad \gamma_{,z} = 2\rho U_{,\rho} U_{,z}$

“axis” on which $\rho = 0$ is regular if, and only if, $\gamma \rightarrow 0$ as $\rho \rightarrow 0$.² If this condition is not satisfied for some value or range of z , then some kind of singularity occurs at these points.

Schwarzschild solution in Weyl coordinates reads

$$e^{2U} = \frac{R_+ + R_- - 2m}{R_+ + R_- + 2m}, \quad e^{2\gamma} = \frac{(R_+ + R_-)^2 - 4m^2}{4R_+ R_-} \quad \text{with} \quad U = \frac{1}{2} \log \left(\frac{R_- + z - m}{R_+ + z + m} \right) \quad R_{\pm}^2 = \rho^2 + (z \pm m)^2$$

which is formally the Newtonian potential for a finite rod, located along the part of the axis $\rho = 0$ for which $|z| < m$, whose mass per unit length is $\sigma = \frac{1}{2}$. Thus the “rod” has length $2m$ and its total mass is m .

Smarr formulas for electrovac spacetimes with line singularities

G.Clement and D.G.,
Phys.Lett. B771 (2017) 457-461
Phys.Lett. B773 (2017) 290-294
Class.Quant.Grav. 35 (2018) no.21, 214002
Phys.Lett. B802 (2020) 135270

- Do Misner strings contribute to the BH entropy?
- Contribution of struts in binary BH solutions to Smarr mass formulas
- Extend to higher dimensions
- Include U(1) vectors and scalar fields
- Describe individual contributions in binary BHs
- Clarify the role of Dirac strings in dyons



Use **Rod structure** analysis to ensure

- Unique description of disconnected components of horizon, struts and MS singular components
- Express individual horizon masses in terms of parameters
- Clarify common and distinct feature of timelike and spacelike rods

Classes of metrics admitting rod structure

1. 4-dim Papapetrou (stationary axisymmetric)

$$ds^2 = -e^{2U}(dt + Ad\phi)^2 + e^{-2U}r^2d\phi^2 + e^{2\nu}(dr^2 + dz^2)$$

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right)U = -\frac{e^{4U}}{2r^2}[(\partial_r A)^2 + (\partial_z A)^2] \quad , \quad \partial_r\left(\frac{e^{4U}}{r}\partial_r A\right) + \partial_z\left(\frac{e^{4U}}{r}\partial_z A\right) = 0 \quad ,$$

Once U and are found, we have to integrate

$$\partial_r \nu = -\partial_r U + r[(\partial_r U)^2 - (\partial_z U)^2] - \frac{e^{4U}}{4r}[(\partial_r A)^2 - (\partial_z A)^2] \quad ,$$

$$\partial_z \nu = -\partial_z U + 2r\partial_r U \partial_z U - \frac{e^{4U}}{2r}\partial_r A \partial_z A \quad .$$

2. Static generalized Weyl metrics in D dim (Emparan and Reall 2002) with D-2 commuting orthogonal Killing vectors

$$ds^2 = -e^{2U_1}dt^2 + \sum_{i=2}^{D-2} e^{2U_i}(dx^i)^2 + e^{2\nu}(dr^2 + dz^2) \quad , \quad \sum_{i=1}^{D-2} U_i = \log r \quad ,$$

with U^i satisfying three-dimensional flat space Laplace equations $\left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2\right)U_i = 0$

Introduce the auxiliary angle γ so that we have the following metric for 3D Euclidean space:

$$dr^2 + r^2 d\gamma^2 + dz^2 = d\sigma_1^2 + d\sigma_2^2 + dz^2$$

$$\sigma_1 = r \cos \gamma, \quad \sigma_2 = r \sin \gamma$$

In 3d vector notation
the Gram matrix
equation reads

$$G^{-1} \vec{\nabla}^2 G = (G^{-1} \vec{\nabla} G)^2$$

Consider now $A=0$, then

$G_{ij}(r, z)$ diagonal matrix in the D-2 space spanned by Killing vectors

$$ds^2 = -e^{2U_1} dx_1^2 + \sum_{i=2}^{D-2} e^{2U_i} dx_i^2 + e^{2\nu} (dr^2 + dz^2)$$

$$\sum_{i=1}^{D-2} U_i = \log r$$

Regularity

$G(r,z)$ smooth for $r > 0 \Rightarrow$ solution regular for $r > 0$

But $\det G(0, z)=0$, so on the polar axis the singularities may be present

Exclude strong curvature singularities $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \rightarrow \infty$ for $r \rightarrow 0$

but allow for delta-function singularities for curvature (struts, Misner strings)

For $r > 0$: $G^{-1}\vec{\nabla}^2 G = (G^{-1}\vec{\nabla} G)^2$ (for the static solutions)

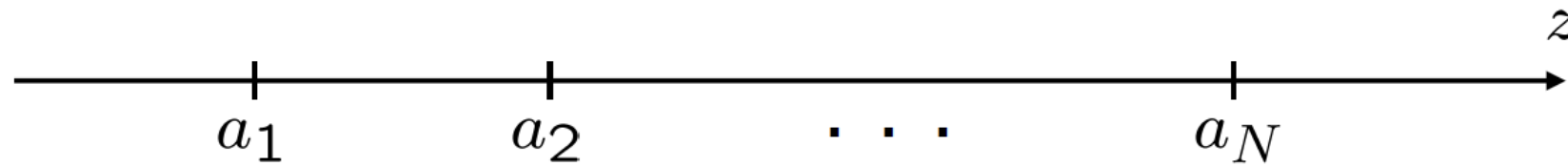
But as $r \rightarrow 0$ Π (eigenvalues of G) = $r^2 \rightarrow 0$

two eigenvalues $\rightarrow 0$ for $r \rightarrow 0 \implies R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \rightarrow \infty$

To avoid this, require $\dim(\ker(G(0,z))) = 1$ except at isolated values of z

(“turning points”) $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_N$ for which $\dim(\ker(G(0,z))) > 1$

Divide z -axis into $N+1$ intervals $[a_{k-1}, a_k]$, $k=1, \dots, N+1$ ($a_0 = -\infty$, $a_{N+1} = \infty$)



We call an interval $[a_{k-1}, a_k]$ a **rod** of the solution $G(r, z)$

$$\dim(\ker(G(0, z))) = 1 \text{ for } a_{k-1} < z < a_k$$

For all $N+1$ rods $[a_{k-1}, a_k]$ we can find a vector such that

$$v_{(k)} = v_{(k)}^i \frac{\partial}{\partial x^i}$$

$$G(0, z)v_{(k)} = 0 \text{ for } z \in]a_{k-1}, a_k[$$

We call $v_{(k)}$ the **direction** of the rod $[a_{k-1}, a_k]$

The **rod-structure** of a solution:

The rods (intervals) $[a_{k-1}, a_k]$ and their directions $v_{(k)}$, $k=1, 2, \dots, N+1$.

For diagonal matrix G (static 4d or generalized Emparan-Reall cases)
all rod directions are orthogonal.

Rod $[a_{k-1}, a_k]$ is a line source for the potential U_i

————→ The reason for calling the intervals $[a_{k-1}, a_k]$ rods

$$G_{ii} = \pm e^{2U_i}$$

NB: Misner strings are rods too

$$\vec{\nabla}^2 U_i = 2\pi \delta^2(\sigma) \rho_i(z)$$

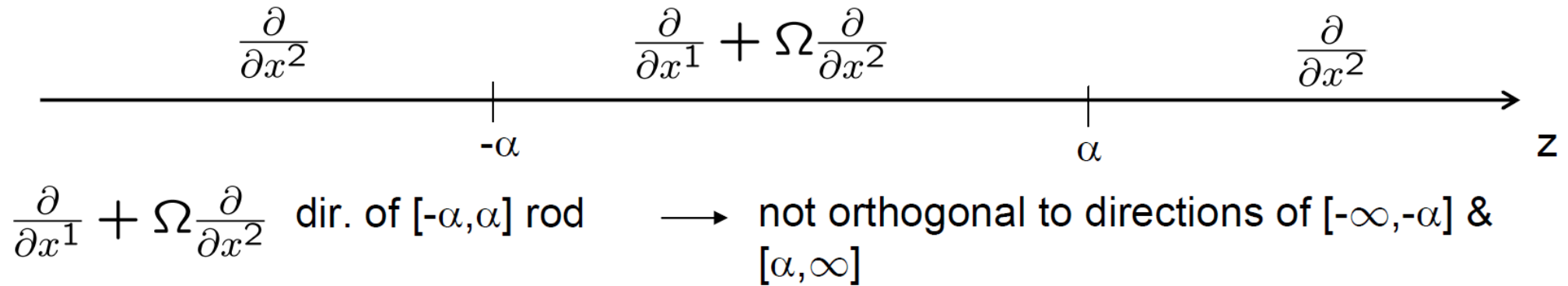
$$\rho_i(z) = \begin{cases} 1 & z \in \text{rod with direction } \frac{\partial}{\partial x^i} \\ 0 & \text{otherwise} \end{cases}$$

In the diagonal case the rod structure is in the one-to-one correspondence with solutions

The same is conjectured for the non-diagonal (stationary) case

Kerr solution

is defined by the following three rods two of which are semi-infinite



Ω : angular velocity of Kerr BH

The rod directions may be time-like, spacelike and null as vectors in D-space

Horizons are finite timelike rods

Rod directions are constant on each chosen rod (cf. constancy of horizon angular velocity)

Finite timelike rods are BH horizons

Rod is eigenvector of Gram matrix with zero eigenvalue $G_{ab}(0, z)l_n^a = 0, \quad z \in [z_n, z_{n+1}]$
Consider D=4 stationary axisymmetric. The direction of a rod defines a Killing vector field l_n^μ of the space-time, written in the basis consisting of $k = \partial_t$ and $m = \partial_\varphi$
Near the rod interior $l_n^2 = G_{ab}l_n^a l_n^b \sim \pm a(z)\rho^2$ and $e^{2\nu} \sim c^2 a(z)$ (c constant)
Thus the quantity $\rho^{-2}e^{-2\nu}l^2$ has a finite limit on the polar axis

Finite timelike rods correspond to black hole horizons, **the associated surface gravity**

$$\kappa_H = (-l_{\mu;\nu}l^{\mu;\nu}/2)^{1/2} = \lim_{\rho \rightarrow 0} \left(-\rho^{-2}e^{-2\nu}G_{ab}l_H^a l_H^b \right)^{1/2}$$

is constant on the axis. The normalized rod direction $l_H = (1/\kappa_H, \Omega_H/\kappa_H)$ contains the angular velocity of the horizon which is also constant on it. Therefore **the normalized rod vector is constant along the rod.** Similarly, the normalized **spacelike rods have constant on them spacelike eigenvectors** $l_n = (1/\kappa_n, \Omega_n/\kappa_n)$ **This defines “spacelike surface gravity” κ_n**

Surface gravity of spacelike rods

$$\kappa_n = \lim_{\rho \rightarrow 0} \left(\rho^{-2} e^{-2\nu} G_{ab} l_n^a l_n^b \right)^{1/2}$$

As we will see later, spacelike rods define distributional singularities on the polar axis, to which belong struts, and Misner strings. Two constant parameters (in 4 dims) are spacelike surface gravity and angular velocity

- The spacelike rod specifies an appropriate spacelike Killing vector in its vicinity which becomes null on the rod itself. This resembles the case of horizon: there the associated Killing vector is timelike and becomes null on the rod
- The difference is that in the horizon case the full spacetime manifold admits continuation into T-region, where the associated Killing vector is spacelike. In the spacelike case no such continuation is possible
- Spacelike rod is associated with conical singularity, unless the Killing coordinate is identified with the period

$$\Delta\eta = 2\pi \lim_{\rho \rightarrow 0} \left(\rho^2 e^{2\nu} \left(G_{ab} l_n^a l_n^b \right)^{-1} \right)^{1/2}$$

Smarr-type mass formulas in presence of distributional sources on the axis

Start with Komar integral for mass at infinity $M = \frac{1}{4\pi} \oint_{\Sigma_\infty} D^\nu k^\mu d\Sigma_{\mu\nu}$ and use Ostrogradski-Gauss to express it over cylindrical surfaces surrounding rods

$M = \sum_n M_n$ where the Einstein-Maxwell equations were used, and Komar forms written in terms of metric

$$M_n = \frac{1}{8\pi} \oint_{\Sigma_n} [g^{ij} g^{ta} \partial_j g_{ta} + 2(A_t F^{it} - A_\varphi F^{i\varphi})] d\Sigma_i$$

Similarly, from the Komar form for angular momentum $J = -\frac{1}{8\pi} \oint_{\Sigma_\infty} D^\nu m^\mu d\Sigma_{\mu\nu}$ we obtain/

$$J = \sum_n J_n, \quad J_n = -\frac{1}{16\pi} \oint_{\Sigma_n} [g^{ij} g^{ta} \partial_j g_{\varphi a} + 4A_\varphi F^{it}] d\Sigma_i$$

Therefore we expressed asymptotical mass and angular momentum in terms of the surface integrals over the rods. Now using the (corrected) Tomimatsu formalism (A. Tomimatsu, Progr. Theor. Phys. **72** (1984) 73) we can pass to one-dimensional integrals along the rods, where the Ernst potentials are used

$$\begin{aligned} M_n &= \frac{1}{4} \int_{z_n}^{z_{n+1}} [\omega \partial_z \text{Im} \mathcal{E} + 2 \partial_z (A_\varphi \text{Im} \psi)] dz & \mathcal{E} &= F - \bar{\psi} \psi + i\chi, & \psi &= v + iu \\ &= \frac{\omega_n}{4} \text{Im} \mathcal{E} \Big|_{z_n}^{z_{n+1}} + \frac{1}{2} (A_\varphi \text{Im} \psi) \Big|_{z_n}^{z_{n+1}} & v &= A_t, & \partial_i u &= F \rho^{-1} \epsilon_{ij} (\partial_j A_\varphi + \omega \partial_j v) \\ & & \partial_i \chi &= -F^2 \rho^{-1} \epsilon_{ij} \partial_j \omega + 2(u \partial_i v - v \partial_i u) \end{aligned}$$

Similarly the angular momentum of a rod is presented as the linear integral which is easily computed

$$J_n = \frac{1}{8} \int_{z_n}^{z_{n+1}} \omega [-2 + \omega \partial_z \text{Im} \mathcal{E} + 2 \partial_z (A_\varphi \text{Im} \psi) - 2 \omega \Phi \partial_z \text{Im} \psi] dz$$

$$= \frac{\omega_n}{4} \left\{ -(z_{n+1} - z_n) + [\omega_n (\text{Im} \mathcal{E} / 2 - \Phi_n \text{Im} \psi) + A_\varphi \text{Im} \psi] \Big|_{z_n}^{z_{n+1}} \right\}$$

and using analogous representation for the electric charge

$$Q_n = \frac{1}{4\pi} \int_{\Sigma_n} \omega \partial_z \text{Im} \psi dz d\varphi = \frac{\omega_n}{2} \text{Im} \psi \Big|_{z_n}^{z_{n+1}}$$

can be finally presented as

$$J_n = \frac{\omega_n}{2} \left(-\frac{z_{n+1} - z_n}{2} + M_n - Q_n \Phi_n \right)$$

Note that the term proportional to the length of a rod can be presented as the product of the rod area and the surface gravity both for timelike and spacelike rods, since

$$\frac{z_{n+1} - z_n}{4} = \frac{\kappa_n}{8\pi} \mathcal{A}_n$$

e.g. for horizon rods $\kappa_H = \sqrt{|e^{-2k}|} / |\omega_H|$, $A_H = \oint d\varphi \int_{z_1}^{z_2} \sqrt{|g_{zz} g_{\varphi\varphi}|} dz = 2\pi \int_{z_1}^{z_2} \sqrt{|e^{2k}|} |\omega| dz$, so that

$$T_H S_H = \frac{\kappa_H}{8\pi} \mathcal{A}_H = \frac{z_2 - z_1}{4}$$

In Misner string case this gives you a choice to interpret this term as mechanical work produced by tension, or as the Misner string entropy (Hawking, Hunter, Carlip, Mann, Bordo, Gray, Hennigar, Kubiznak...)

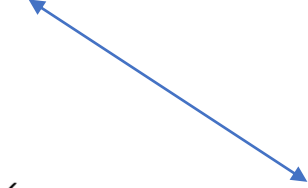
For the individual black hole (this works for binary BH)
 There is no contributions from magnetic charges or
 NUT charges, these notions are global because of
 Dirac and Misner strings

$$M_H = 2\Omega_H J_H + 2T_H S_H + \Phi_H Q_H$$

The string masses can be presented as $M_n = 2\Omega_n J_n + \frac{1}{2}L_n + \Phi_n Q_n$

The asymptotic mass, on the contrary, contains string
 terms in terms of spacelike rods

$$L_n = z_{n+1} - z_n$$

$$M = \sum_{H_n} \left(2\Omega_{H_n} J_{H_n} + \frac{\kappa_{H_n} \mathcal{A}_{H_n}}{4\pi} + \Phi_{H_n} Q_{H_n} \right) + \sum_{S_n} \left(2\Omega_{S_n} J_{S_n} + \frac{\kappa_{S_n} \mathcal{A}_{S_n}}{4\pi} + \Phi_{S_n} Q_{S_n} \right)$$


where the string surface terms were presented in the “entropic” form. Note, that for the infinite rods the corresponding length should be cut off. To cancel infinities, the infinite Misner and Dirac string should be arranged in a symmetric way under North/South interchange.

This construction is applicable to multicenter axially symmetric solutions with struts ensuring force balance needed for equilibrium. It follows that masse and angular momenta are additive quantities, which may look surprising since the bulk electromagnetic contributions are taken into account. These are already included in the charge terms

Dirac strings (DS) gravitate

- Usually regarded as artefact of magnetic monopole vector potential
- When gravity is switched on, DS turns out to gravitate, i.e., DS is heavy
- Illustrated by difference between the horizon mass Smarr and total mass Smarr for dyonic Kerr-Newman

$$ds^2 = -F(dt - \omega d\varphi)^2 + F^{-1}[e^{2k}(d\rho^2 + dz^2) + \rho^2 d\varphi^2]$$

$$F = \frac{f}{\Sigma}, \quad e^{2k} = \frac{f}{\sigma^2(x^2 - y^2)},$$

Weyl coordinates by $\rho = \sigma(x^2 - 1)^{1/2}(1 - y^2)^{1/2}$, $z = \sigma xy$
prolate spheroidal coordinates $x \geq 1$, $y \in [-1, +1]$

$$f = \sigma^2(x^2 - 1) - a^2(1 - y^2), \quad \Sigma = (\sigma x + M)^2 + a^2 y^2, \quad \sigma^2 = M^2 - Q^2 - P^2 - a^2$$

$$\omega = -a(1 - y^2) \frac{2M(\sigma x + M) - Q^2 - P^2}{f},$$

$$A_t = \frac{-Q(\sigma x + M) + aPy}{\Sigma}, \quad A_\varphi = -Py - C - aA_t(1 - y^2) \quad \Omega_H = \frac{a}{(M + \sigma)^2 + a^2} = \frac{a}{\Sigma_0}$$

Horizon Smarr $M_H = 2\Omega_H J_H + 2T_H S + \Phi_H Q_H$ does not contain magnetic charge, while total mass does

$M = 2\Omega_H J + 2T_H S + \Phi_H Q + \tilde{\Phi}_H P$, where magnetic potential is $\tilde{\Phi}_H = P(M + \sigma)/\Sigma_0$. The difference is due to the Dirac strings given by the non-zero surface integral around the string:

$$M_{S_\pm} = \mp \frac{1}{8\pi} \int_{\Sigma_\pm} \sqrt{|g|} g^{yy} \left[g^{ta} \partial_y g_{ta} + 2(g^{ta} A_t \partial_y A_a - g^{\varphi a} A_\varphi \partial_y A_a) \right] dx d\varphi$$

$$M_{S_+} + M_{S_-} = \frac{P^2(M + \sigma)}{\Sigma_0}$$

This is valid only for symmetric gauge for DS (equal North and South segments)

Kerr-NUT

In Boyer-Lindquist
Coordinates

$$ds^2 = -\frac{\Delta}{\Sigma} (dt + P_\theta d\varphi)^2 + \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2 \theta}{\Sigma} (adt - P_r d\varphi)^2,$$

$$P_\theta = 2n \cos \theta + 2s - a \sin^2 \theta, \quad P_r = r^2 + a^2 + n^2 - 2as,$$

$$\Sigma = P_r + aP_\theta = r^2 + (n + a \cos \theta)^2, \quad \Delta = r^2 - 2mr + a^2 - n^2,$$

Passing to Weyl
coordinates

$$\rho = \sqrt{\Delta} \sin \theta, \quad z = (r - m) \cos \theta \quad \text{one finds three rods}$$

$(-\infty, -z_H], [-z_H, z_H], [z_H, \infty)$ with $z_H = r_H - m$, $r_H = m + \sqrt{m^2 + n^2 - a^2}$,
joining pairwise at $z = \pm z_H$ and the directions with the parameters

$$\kappa_H = \frac{r_H - m}{(mr_H + n^2 - as)}, \quad \Omega_H = \frac{a\kappa_H}{2(r_H - m)}, \quad \kappa_\pm = \frac{1}{2(n \pm s)}, \quad \Omega_\pm = \mp \kappa_\pm.$$

The rod l_H defines the spacetime Killing vector $\xi_H = \partial_t + \Omega_H \partial_\varphi$ which is timelike outside the horizon, and becomes null on it. The rods l_\pm define the Killing vectors $\xi_\pm = \partial_t + \Omega_\pm \partial_\varphi$ which are *spacelike* outside the polar axis for $|z| > z_H$ and become null on the Misner strings. Their norm in the vicinity of the polar axis for $s = 0$ is

$$\xi_\pm^2 = \frac{r^2 + (n \pm a)^2}{4n^2} \sin^2 \theta$$

Obviously it vanishes on the polar axis

The associated “surface gravity” κ_\pm , therefore, is not associated with particle acceleration, neither with a redshift factor, so it can hardly be interpreted as Hawking temperature.

$$M_H = \frac{\omega_H}{4} (\chi_+ - \chi_-) = \frac{a\omega_H}{2r_H} = m + \frac{n^2}{r_H}$$

$$J_H = \frac{\omega_H}{2} \left(M_H - \frac{L_H}{2} \right) = \frac{a^2\omega_H}{2r_H} = aM_H,$$

$$M_{\pm} = \mp\omega_{\pm}\chi_{\pm} = -\frac{n(n \mp a)}{2r_H},$$

$$J_{\pm} = \frac{\omega_{\pm}}{2} \left(M_{\pm} - \frac{L_{\pm}}{2} \right),$$

where $\omega_{\pm} = \mp 2n$, $L_H = 2\sigma$ and $L_{\pm} = R - \sigma$, with R a regularization length of the infinite rods ($R \rightarrow \infty$). For our symmetrical setting, $s = 0$, the sum of the string angular momenta is finite:

$$J_+ + J_- = -n(M_+ - M_-) = -\frac{an^2}{r_H} = a(M_+ + M_-) \quad J = J_H + J_+ + J_- = a(M_H + M_+ + M_-) = aM$$

where the total mass M has the value m . Note also that the strings are always rotating in opposite directions, $\Omega_{\pm} = \mp 1/(2n)$, even in the case where the horizon is non-rotating, $a = 0$. But in this case the sum of their angular momenta is zero.

Note that the Kerr proportionality holds separately for the horizon rod, and for the sum of the strings, as well as for the global quantities

Finally, we can write Smarr mass in “mechanical” terms:

$$M = 2T_H S_H + 2\Omega_H J_H + 2\Omega_+ \tilde{J}_+ + 2\Omega_- \tilde{J}_-$$

where

$$\tilde{J}_{\pm} \equiv J_{\pm} + \frac{\omega_{\pm} L_{\pm}}{4} = \frac{\omega_{\pm} M_{\pm}}{2} = \pm \frac{n^2(n \mp a)}{2r_H}$$

Einstein-Maxwell dilaton

The bulk term now contain the scalar field but still may be reduced to the boundary term, so the mass additivity still holds

The only difference with the EM case is due to dilaton exponent in the rod charges

We therefore again obtain

$$M = \sum_n M_n, \quad M_n = 2T_n S_n + 2\Omega_n J_n + \Phi_n Q_n$$

$$\mathcal{S} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(R - 2(\partial\phi)^2 - e^{-2\alpha\phi} F^2 \right)$$

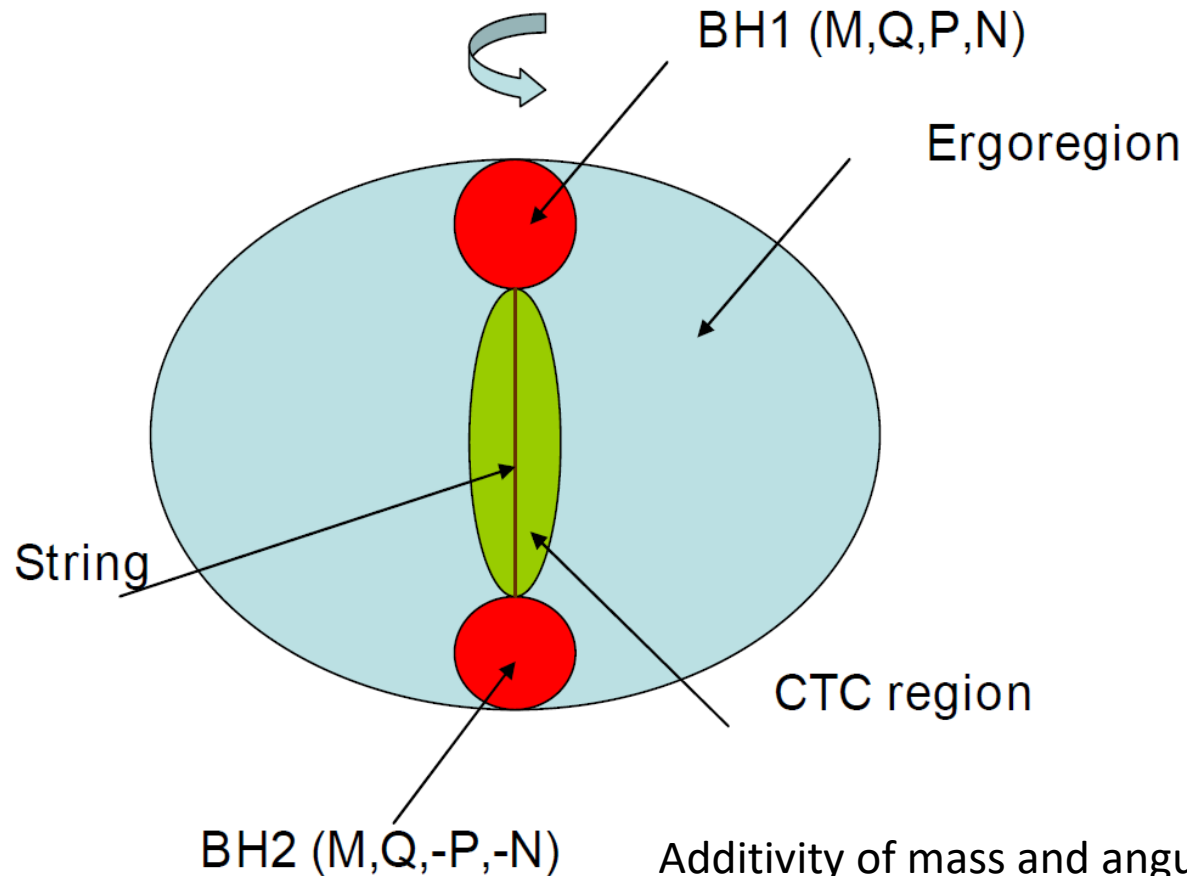
$$M = \sum_n \frac{1}{4\pi} \oint_{\Sigma_n} \nabla^\nu k^\mu d\Sigma_{\mu\nu} + M_E$$

$$M_E = \sum_n \frac{1}{4\pi} \int_{\Sigma_n} d\Sigma_{ti} \left\{ e^{-2\alpha\phi} (F^{it} A_t - F^{i\varphi} A_\varphi) \right\}$$

$$Q_n = \frac{1}{4\pi} \oint_{\Sigma_n} e^{-2\alpha\phi} F^{it} d\Sigma_{ti}$$

Example of Binary BH with Kerr asymptotic

G.Clement and D.G.,
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Additivity of mass and angular momentum.
The sum of electric charges is canceled by the
charge of the string

Each BH has mass, electric and magnetic charges and NUT charges: masses and electric charges are equal, magnetic charges and NUT parameters are opposite. Such configuration has AF (Kerr) asymptotic of rotating vacuum black hole, though it has a complicated internal structure inside the ergosphere. So it may seem to violate uniqueness, but in fact it does not because of singular string between two BHs. The string is superposition of a structure (rotating and charged “cosmic string”), Dirac and Misner strings

$$M = M_+ + M_- + M_S$$

$$J = J_+ + J_- + J_S$$

Conclusions

- Struts and Misner strings do contribute to asymptotic mass Smarr formulas
- The surface terms in Smarr formula for spacelike terms can be presented either in the entropic form, or in mechanical terms, as work done by tension. Which interpretation is correct still has to be discussed. Entropic interpretation is favored by analogy with BH as Killing horizon, but there is not information loss nor the internal region of defect.
- Horizon mass Smarr formulas do not contain contribution of magnetic and NUT charges, reflecting their global nature. Asymptotic mass Smarr formulas do contain contributions of Dirac and Misner strings or, in other parametrization, of magnetic and NUT charges. Dirac strings are heavy. Only symmetric North/South configuration is consistent for both Dirac and Misner strings
- Total mass, charges and angular momenta of aligned multi BH solutions with Kerr asymptotic can be expressed additively as sums of analogous individual parameters of all constituents including BHs, struts and Misner and Dirac strings

Thanks! Happy research!

