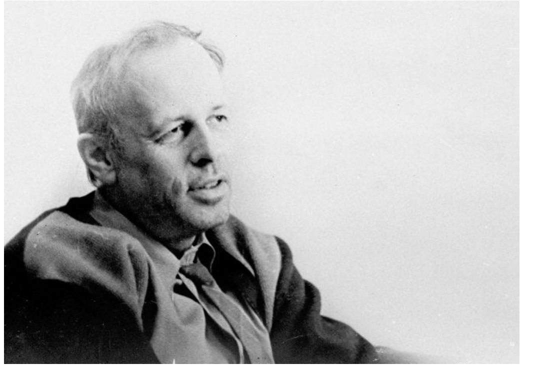
International workshop "Quantum Gravity and Cosmology"
(dedicated to A.D. Sakharov's centennial)
June the 8th 2021



# Domain walls without a potential

**I. Basics of, and, new look on « canonical » domain walls**

**I.1. Field configuration**

**I.2. Energy, stability and vacuum topology**

**II. Non standard scalar theories and « k-defects »**

**III. Domains walls without a potential**

**III.1. Model building**

**III.2. Understanding the models**

**III.3. Issues, perspectives and conclusions**


Cédric Deffayet
(IAP and IHÉS, CNRS Paris)



**C.D., F. Larrouturou**
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## I.1. Domain wall field configurations

Well known solutions of simple
scalar field theories with a potential

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Well known solutions of simple scalar field theories with a potential

*i.e.* theories with a Lagrangian of the form

$$\mathcal{L}_{\rm can}(\phi, X) = X - V(\phi)$$

With

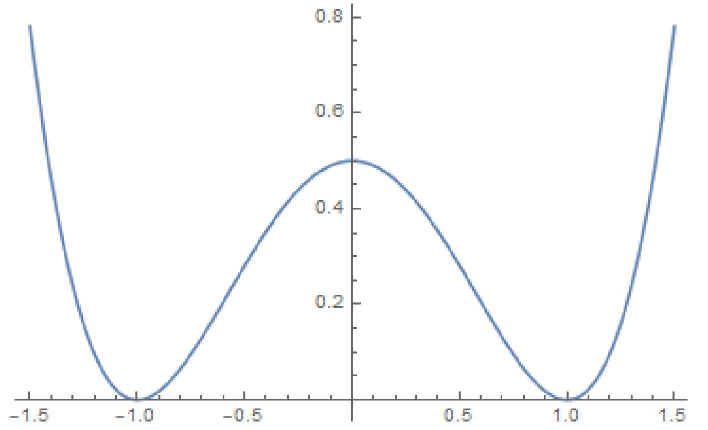
$$X = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,$$

$V(\phi)$ A potential with two or more minima at different values $\phi^k_{min}$ of the field, such that $V(\phi)$ has the same value at these minima

Standard exemples of potentials admitting domain walls

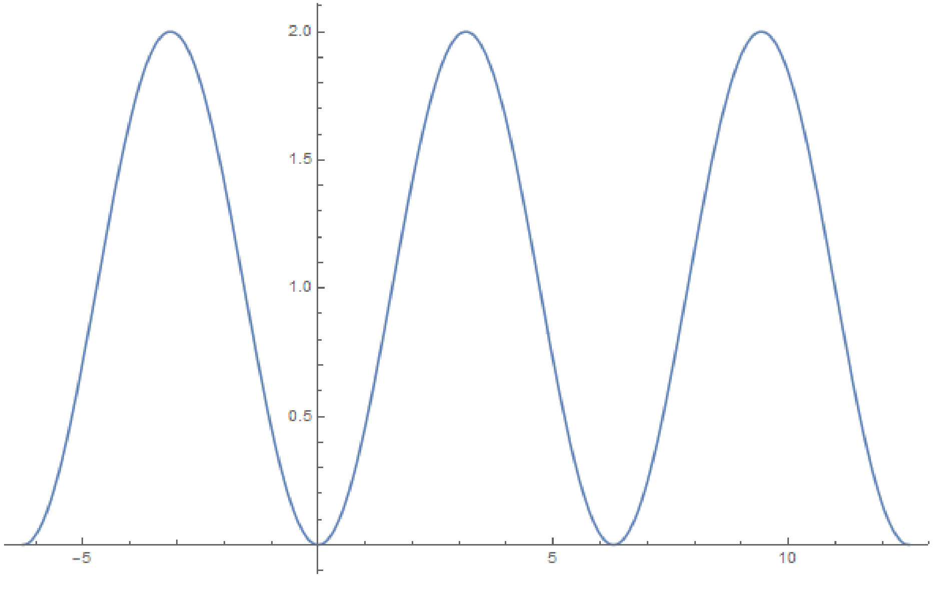
« Mexican hat model »

$$V_{mh} = \frac{1}{2}\left(1 - \phi^2\right)^2$$




« Sine-Gordon model »

$$V_{sG} = 1 - \cos\left(\phi\right)$$



Domain walls,

most easily discussed in 1(t coordinate) +1(z coordinate) dimensions where they are usually called « kinks »,

are solutions interpolating between two adjacent vaccua between $z = -\infty$ and $z = +\infty$

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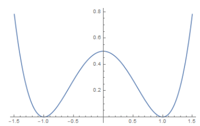
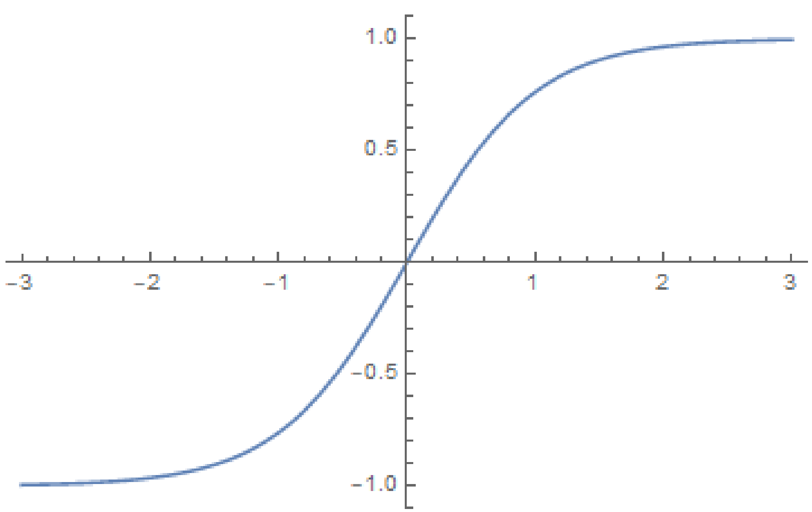


$$\phi_{mh}(z) = \pm \tanh(z)$$

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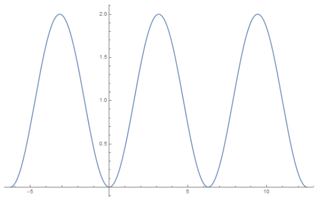
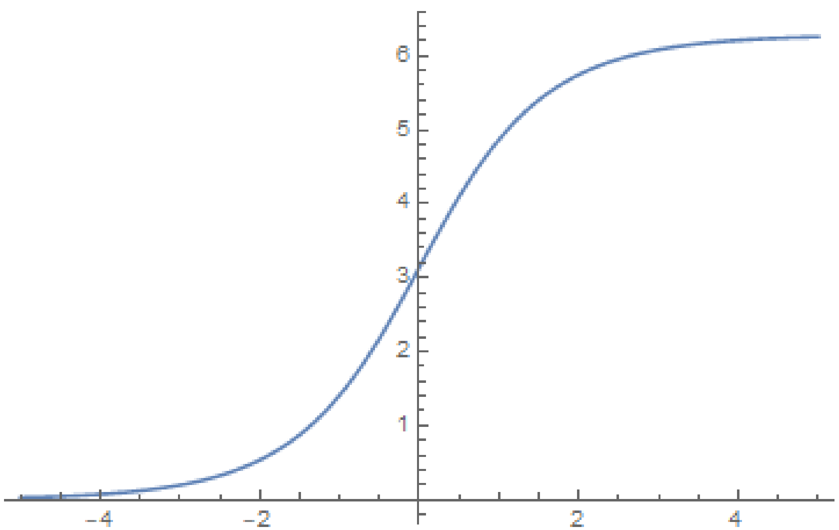


$$\phi_{sG}(z) = 2\pi k + 4 \arctan e^{z}$$



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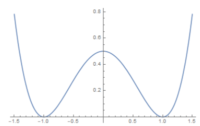

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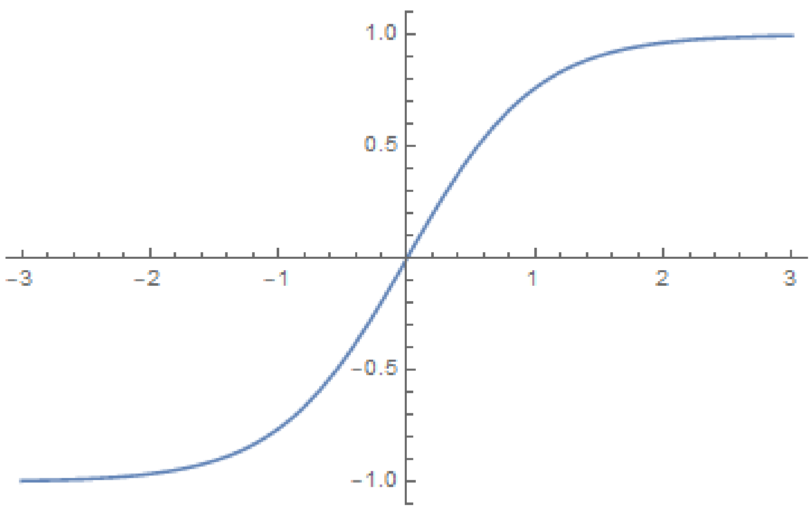


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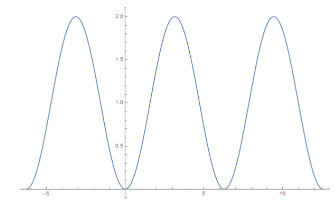

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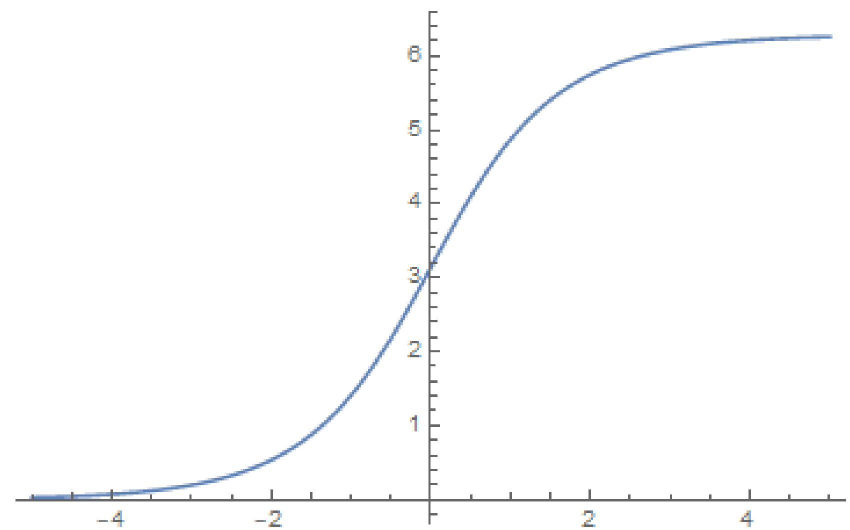


Very similar looking solutions !

The similarity can be made more explicit using the change of variable

Mexican hat model

$$\phi = \tanh(\psi) \Leftrightarrow \psi = \tanh^{-1} \phi$$

the canonical Mexican hat Lagrangian is transformed into

$$\mathcal{L}_{\mathrm{mh}}(\psi, X_\psi) = \frac{X_\psi - \frac{1}{2}}{\cosh^4(\psi)}$$

Sine-Gordon model

$$\phi(\psi) = 2\pi k + 4 \arctan e^{\psi}$$

$$\Leftrightarrow \psi = (-1)^k \ln \left| \tan\left(\frac{\phi}{4}\right) \right|$$

the canonical Sine-Gordon Lagrangian is transformed into

$$\mathcal{L}_{\mathrm{sG}}(\psi, X_\psi) = \frac{4}{\cosh^2(\psi)} \left( X_\psi - \frac{1}{2} \right)$$

Such that the domain wall solutions read in both cases

$$\psi = \pm z$$

( NB: in both cases one has $d\psi = \pm \dfrac{d\phi}{\sqrt{2V}}$ )

Rather obvious generalization:

some constant

$$\mathcal{L}_{\mathrm{k,can}}(\psi, X_\psi) = \frac{\mathcal{K}}{\cosh^{2k}(\psi)}\left(X_\psi - \frac{1}{2}\right)$$

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$$\equiv \; v(\psi)w(X_\psi)$$

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Such as the part of the field equation which is **<u>not proportional to second derivatives</u>** is simply

$$v'(\psi)\left(2X_\psi w'(X_\psi) - w(X_\psi)\right)$$

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Hence a field profile of the form $\psi \; = \; \lambda z$ (yielding $X_\psi = -\lambda^2/2$)

Is a solution, provided $-\lambda^2/2$ is a root of the function $y$ defined by

$$y(X_\psi) = 2X_\psi w'(X_\psi) - w(X_\psi)$$

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For $w(X_\psi)\ =\ 2X_\psi\ -\ 1$ one has $y(X_\psi)\ =\ 2X_\psi\ +\ 1$

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For $w(X_\psi) = 2X_\psi - 1$ one has $y(X_\psi) = 2X_\psi + 1$ ✓

Domain walls solutions have also finite energy

Hence from the generic $\mathcal{L}\left(\psi, X_\psi\right) \equiv v(\psi)w(X_\psi)$

We need the total energy of the wall …

$$\mathcal{H}_{dw} = 2\int_{-\infty}^{+\infty} v(\psi)d\psi$$

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For the canonical (generalized) models $\mathcal{L}_{\mathrm{k,can}}(\psi, X_\psi) = \dfrac{\mathcal{K}}{\cosh^{2k}(\psi)}\left(X_\psi - \dfrac{1}{2}\right)$

We get

$$\begin{cases} \mathcal{H}_{\mathrm{k,can}} = \mathcal{K}\displaystyle\int_{-\infty}^{+\infty}\frac{\mathrm{d}z}{\cosh^{2k} z} = \mathcal{K}\,\mathcal{I}_k \\ \mathcal{I}_k = \displaystyle\int_{-\infty}^{+\infty}\frac{dz}{\cosh^{2k}(z)} = \frac{\sqrt{\pi}\,\Gamma(k)}{\Gamma(k+1/2)} \quad \text{for} \quad k \geq \frac{1}{2} \end{cases}$$

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For a theory in the canonical class $\mathcal{L}_{\rm can}(\phi, X) = X - V(\phi)$

The energy of an arbitrary field configuration can be rewritten using the « Bogomolny trick » as

time derivative

z derivative

$$\mathcal{H}(t) = \int dz \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\phi' \pm \sqrt{2V}\right)^2 \mp \sqrt{2V}\phi'\right]$$

Bogomolny, 1976

I.e. $\mathcal{H}(t) = \mathcal{H}_{kin}(t) + \mathcal{H}_{grad}(t) + \mathcal{H}_{\infty}(t)$

With
$$\begin{cases} \mathcal{H}_{kin}(t) = \int \frac{1}{2}\dot{\phi}^2 dz \\ \mathcal{H}_{grad}(t) = \int \frac{1}{2}\left(\phi' \pm \sqrt{2V}\right)^2 dz \\ \mathcal{H}_{\infty}(t) = \int \mp\sqrt{2V}\phi' dz \end{cases}$$

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Vanish for domain wall configurations

With
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« Topological » term

The topological term $\mathcal{H}_\infty(t) = \int \mp\sqrt{2V}\phi' dz$

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2D Levi-Civita tensor

It can be linked to the trivially conserved current $J^\mu = \mathcal{C}\epsilon^{\mu\nu}\partial_\nu\phi$

yielding the « topological charge »

$$Q = \int_{z=-\infty}^{+\infty} dz\, J^0(z) = \mathcal{C}\left(\phi(+\infty) - \phi(-\infty)\right)$$

$\phi$

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$$Q = \int_{z=-\infty}^{+\infty} dz J^0(z) = \mathcal{C}\left(\phi(+\infty) - \phi(-\infty)\right)$$

But any function of $\phi$ Yields such a conserved current

E.g. the choice $\tilde{J}^\mu = \tilde{\mathcal{C}}\epsilon^{\mu\nu}\partial_\nu\left(\int_{\phi_0}^{\phi}\sqrt{2V(u)}du\right)$

yields the conserved charge $\tilde{Q} = \int_{z=-\infty}^{+\infty} dz \tilde{J}^0(z) = \mp\tilde{\mathcal{C}}\mathcal{H}_\infty$

The stability of the domain wall configuration is guaranteed by the decomposition

$$\mathcal{H}(t) = \mathcal{H}_{kin}(t) + \mathcal{H}_{grad}(t) + \mathcal{H}_{\infty}(t)$$

$$\begin{cases} \mathcal{H}_{kin}(t) &= \displaystyle\int \frac{1}{2}\dot{\phi}^2 dz \\ \mathcal{H}_{grad}(t) &= \displaystyle\int \frac{1}{2}\left(\phi' \pm \sqrt{2V}\right)^2 dz \\ \mathcal{H}_{\infty}(t) &= \displaystyle\int \mp\sqrt{2V}\phi' dz \end{cases}$$

Indeed, any local perturbation of the wall, with the same « topological charge » $\mathcal{H}_{\infty}$

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Indeed, any local perturbation of the wall, with the same « topological charge » $\mathcal{H}_{\infty}$ has an energy larger (or equal) to the one of the wall (also $\mathcal{H}_{\infty}$ )

NB: this argument is unrelated to the « topology » of the vacuum manifold being labeled by the disconected values $\phi^k_{min}$

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Decomposing such a perturbation $\varphi$ as $\varphi = \sum \varphi_k(z) e^{i\omega_k t}$

It obeys the mode equation of motion

$$\left(\mathcal{Z}^{zz}\varphi_k'\right)' - \left(\mathcal{Z}^{00}\,\omega_k^2 + \mathcal{M}^2\right)\varphi_k = 0$$

with $\begin{cases} -\mathcal{Z}^{00}\mathcal{Z}^{zz} > 0 \\ 0 < 2\int \mathrm{d}z \mathcal{Z}^{00} X < +\infty \end{cases}$

And e.g. $\begin{cases} \left.\begin{array}{l} \mathcal{Z}^{zz} = -\mathcal{Z}^{00} \;=\; 1 \\ \mathcal{M}^2 = 2\left(3\phi^2 - 1\right) \;\approx\; 6\tanh^2(z) - 2 \end{array}\right\} \text{Mexican hat} \\ \left.\begin{array}{l} \mathcal{Z}^{zz} = -\mathcal{Z}^{00} \;=\; 1 \\ \mathcal{M}^2 = \cos(\phi) \;\approx\; 2\tanh^2(z) - 1 \end{array}\right\} \text{Sine-Gordon} \end{cases}$

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✔ Stable perturbations (including always the zero mode $\varphi_0 \propto \phi'$)

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# II. Non standard scalar theories and « k-defects »

The recent years have seen a renewal of interest in scalar(-tensor) theories with non trivial kinetic term for the scalar

K-essence $$\mathcal{L} = P\left(\phi, X = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi\right)$$

Bekenstein, Milgrom 1984
Armendariz-Picon, Damour, Mukhanov 1999
Armendariz-Picon, Mukhanov Steinhardt, 2000-2001

DGP decoupling limit, flat and curved space time Galileons

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi\Box\phi \quad \ldots$$

Luty, Porrati, Rattazzi 2003
Nicolis, Rattazzi, Trincherini, 2008
C.D., Esposito-Farese, Vikman 2009

Horndeski and beyond

Horndeski 1974
C.D., Gao, Steer, Zahariade 2011
Zumalacarregui, Garcia-Bellido 2014
Gleyzes, Langlois, Piazza, Vernizzi, 2015...

For this talk, a related (and older) construction is that of the
Skyrme model of Baryons as solitons (here with a quartet of scalar fields $\phi^a$)

$$\mathcal{L} = \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2} \left(\partial_\mu \phi^a \partial^\mu \phi^a\right)^2 + \frac{1}{2} \left(\partial_\mu \phi^a \partial^\mu \phi^a\right) \left(\partial_\mu \phi^b \partial^\mu \phi^b\right) - V(\phi^a)$$

Skyrme 1962

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Skyrme 1962

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Several worked studied « k-defects » (Babichev 2006) in this renewed context

Babichev; Sarangi; Bazeia, Losano, Menezes, Oliveira; Jin, Li, Liu; Adam, Sanchez-Guillen, Wereszcynski; Bazeia, Lobao, Menezes; Chagoya, Tasinato; Andrews, Lewandowski, Trodden, Wesley….

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The existence of these k-defects all rely on a potential with a non trivial vacuum topology

# III. Domain walls without a potential

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## III.1. model building

Can one builds defects out of non standard theories
which are not « k-defects » *i.e.* have no potential ?

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Consider k-essence models $\mathcal{L} = P\left(\phi, X\right)$

(with $\mathcal{L}$ not of the trivial form $\mathcal{L}_{\rm can}(\phi, X) = X - V(\phi)$ )

Hyperbolic, with positive energy

Standard conditions in order to get a consistent theory:

Armendariz-Picon, Damour, Mukhanov 1999

$$\left\{ \begin{array}{lcl} 0 & < & P_X \\ 0 & < & 2XP_{XX} + P_X \end{array} \right.$$

Where here and henceforth:
$P_X = \partial P(\phi, X)/\partial X$

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Where here and henceforth: $P_X = \partial P(\phi, X)/\partial X$

Then look for a stable domain wall configuration $\phi(z)$

Perturbations $\varphi(t, z)$ around an arbitrary such field configuration have the Lagrangian

$$\delta^{(2)}\mathcal{L} = -\frac{1}{2}\left[\mathcal{Z}^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + \mathcal{M}^2\varphi^2\right]$$

With $$\begin{cases} \mathcal{Z}^{00} = -P_X \\ \mathcal{Z}^{zz} = \mathcal{J}_X = 2XP_{XX} + P_X \\ \mathcal{M}^2 = -\mathcal{E}_\phi = \mathcal{J}_{\phi\phi} - \mathcal{J}_{\phi X}\,\phi'' \end{cases}$$

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$$\delta^{(2)}\mathcal{L} = -\frac{1}{2}\left[\mathcal{Z}^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi + \mathcal{M}^2\varphi^2\right]$$

With
$$\left\{ \begin{array}{l} \boxed{\begin{array}{l} \mathcal{Z}^{00} = -P_X \\ \mathcal{Z}^{zz} = \mathcal{J}_X = 2XP_{XX} + P_X \end{array}} \\ \mathcal{M}^2 = -\mathcal{E}_\phi = \mathcal{J}_{\phi\phi} - \mathcal{J}_{\phi X}\,\phi'' \end{array} \right.$$

We assume that $P(\phi, X)$ can be power expanded as

$$P(\phi, X) = \sum_{n \geq 2} \alpha_n(\phi)\,(-2X)^{n/2}$$

(such that $\alpha_0$ is set to zero, to avoid a potential)

For a field configuration $\phi(z)$, the field equations have the first integral

$$\mathcal{J} = 2X\,P_X - P \approx \mathcal{J}_0$$

constant
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$\Downarrow$

✗ Does not admit a domain wall solution

Separable theories ?

$$P(\phi, X) = \alpha(\phi) \sum_{n \geq 2} \beta_n \left(-2X\right)^{n/2}$$

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E.g. looking for theories accomodating an *tanh* profile (i.e. the domain wall profile of the mexican hat theory)

we have that, on shell $\phi = \tanh(z)$

Implying the on shell functional relations $\begin{cases} X \approx -f^2/2 \\ f(z) \approx 1 - \phi^2(z) \end{cases}$

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Implying the on shell functional relations $\begin{cases} X \approx -f^2/2 \\ f(z) \approx 1 - \phi^2(z) \end{cases}$

This can be used to « reconstruct » the theory admitting such a profile integrating the relation

$$\mathcal{J} = 2X\, P_X - P \approx \mathcal{J}_0$$

One such family of theories has the Lagrangians

$$P(\phi, X) = X + \frac{\beta_{n,p}}{2(n-1)} \left(1 - \phi^2\right)^p (-2X)^{n/2} + \frac{\beta_{m,q}}{2(m-1)} \left(1 - \phi^2\right)^q (-2X)^{m/2}$$

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Finiteness of the energy implies

Constant

$$P_{n,m}(\phi, X) = X + \frac{1-\kappa}{2(n-1)} \frac{\left(-2X\right)^{n/2}}{\left(1-\phi^2\right)^{n-2}} + \frac{\kappa}{2(m-1)} \frac{\left(-2X\right)^{m/2}}{\left(1-\phi^2\right)^{m-2}}$$

Yielding a $\phi = \tanh(z)$ domain wall with total energy

$$\mathcal{H}(z) = \frac{1}{2}\left(\frac{n-2}{n-1} + \frac{(m-n)\kappa}{(n-1)(m-1)}\right)\cosh^{-4}(z)$$

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Positivity of the energy implies

$$\frac{n-m}{n-2}\kappa < m-1$$

Stability of perturbations further restricts

$$n > 2, \qquad m > 2, \qquad 1 < \frac{n-m}{n-2}\kappa < m-1$$

So, to summarize, the k-essence theories

$$\begin{cases} P_{n,m}(\phi, X) = X + \dfrac{1-\kappa}{2(n-1)} \dfrac{(-2X)^{n/2}}{(1-\phi^2)^{n-2}} + \dfrac{\kappa}{2(m-1)} \dfrac{(-2X)^{m/2}}{(1-\phi^2)^{m-2}} \\ n > 2, \qquad m > 2, \qquad 1 < \dfrac{n-m}{n-2}\kappa < m-1 \end{cases}$$

Have stable domain walls with (exactly) the mexican hat model profile $\phi = \tanh(z)$ ,

stable perturbations, and no potential

in the sense that $P\left(\phi, X=0\right)$

One of the simplest (an interesting – see thereafter) such model is obtained by

$$P(\phi, X) = X + \frac{3\,X^2}{2\left(1-\phi^2\right)^2} + \frac{X^3}{\left(1-\phi^2\right)^4}$$

Which has $(n, m) = (4, 6)$ and $\kappa = -5/4$

# III. Domain walls without a potential

## III.2. Understanding the obtained models

### III.2.1. Changing variables

$$P_{n,m}(\phi, X) = X + \frac{1-\kappa}{2(n-1)} \frac{(-2X)^{n/2}}{(1-\phi^2)^{n-2}} + \frac{\kappa}{2(m-1)} \frac{(-2X)^{m/2}}{(1-\phi^2)^{m-2}}$$

Using $\phi = \tanh(\psi) \Leftrightarrow \psi = \tanh^{-1}\phi$

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Using $\phi = \tanh(\psi) \Leftrightarrow \psi = \tanh^{-1}\phi$

$$P_{n,m}(\psi, X_\psi) = \frac{1}{\cosh^4\psi}\left(X_\psi + \frac{1-\kappa}{2(n-1)}(-2X_\psi)^{n/2} + \frac{\kappa}{2(m-1)}(-2X_\psi)^{m/2}\right)$$

This belongs to the family of theories of the form

$$\mathcal{L} = \left(\sum_{n\in\mathbb{N}} \kappa_n (-2X_\psi)^{n/2}\right)\cosh^{-4}\psi$$

Constants

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Among which the mexican hat theory

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Constants

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$$\mathcal{L}_{\mathrm{mh}}(\psi, X_\psi) = \frac{X_\psi - \frac{1}{2}}{\cosh^4(\psi)}$$

For suitable $\kappa_n$ , $\psi = \pm z$ is a (domain wall) solution of the field equation

One unpleasant aspect of …

$$P_{n,m}(\phi, X) = X + \frac{1-\kappa}{2(n-1)}\frac{(-2X)^{n/2}}{(1-\phi^2)^{n-2}} + \frac{\kappa}{2(m-1)}\frac{(-2X)^{m/2}}{(1-\phi^2)^{m-2}}$$

… is the singularity at $\phi = \pm 1$

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Changing variable $\mathrm{d}\xi_p \equiv \dfrac{\mathrm{d}\phi}{(1-\phi^2)^{1-\frac{2}{p}}}$ for $p - 2 \in \mathbb{N}$

Yielding $\xi_p(\phi) = {}_2F_1\left[\frac{1}{2}, 1 - \frac{2}{p}; \frac{3}{2}; \phi^2\right] \phi$

Gauss hypergeometric function

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For $p = m$

$$P = \frac{\kappa\,(-2X_\xi)^{m/2}}{2(m-1)} + \frac{1-\kappa}{2(n-1)}\left(1-\phi^2\right)^{2\left(1-\frac{n}{m}\right)}(-2X_\xi)^{n/2} + \left(1-\phi^2\right)^{2\left(1-\frac{2}{m}\right)}X_\xi$$

(where $\phi = \phi(\xi)$ )

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(where $\phi = \phi(\xi)$ )

No more singular at $\phi = \pm 1$ … but non standard kinetic terms

**More on this change of variables** $\xi_p(\phi) = {}_2F_1\left[\frac{1}{2}, 1-\frac{2}{p}; \frac{3}{2}; \phi^2\right]\phi$

## More on this change of variables

$$\xi_p(\phi) = {}_2F_1\left[\frac{1}{2}, 1-\frac{2}{p}; \frac{3}{2}; \phi^2\right]\phi$$

Some special values of p lead to nicer forms

Elliptic integral of the first kind

$$\xi_2 = \phi \qquad \xi_8 = 2F\left(\frac{\arcsin(\phi)}{2}, \sqrt{2}\right)$$

$$\xi_4 = \arcsin(\phi) \qquad \xi_\infty = \tanh^{-1}\phi$$

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Maps $\phi = \pm 1$ to the finite $\xi_p^\pm \equiv \xi_p\,(\phi = \pm 1) = \pm\frac{\sqrt{\pi}\,\Gamma\left(\frac{2}{p}\right)}{2\,\Gamma\left(\frac{1}{2}+\frac{2}{p}\right)}$

with diverging $d\xi_b/d\phi$ at $\phi = \pm 1$

The inverse mapping $\phi = \phi(\xi_p)$

can be naturally extended on the whole real line for $\xi_p$

E.g. $\phi = \sin(\xi_4)$ or $\phi = \sin\left(2\mathrm{am}(\xi_8/2)\right))$

So that the model …

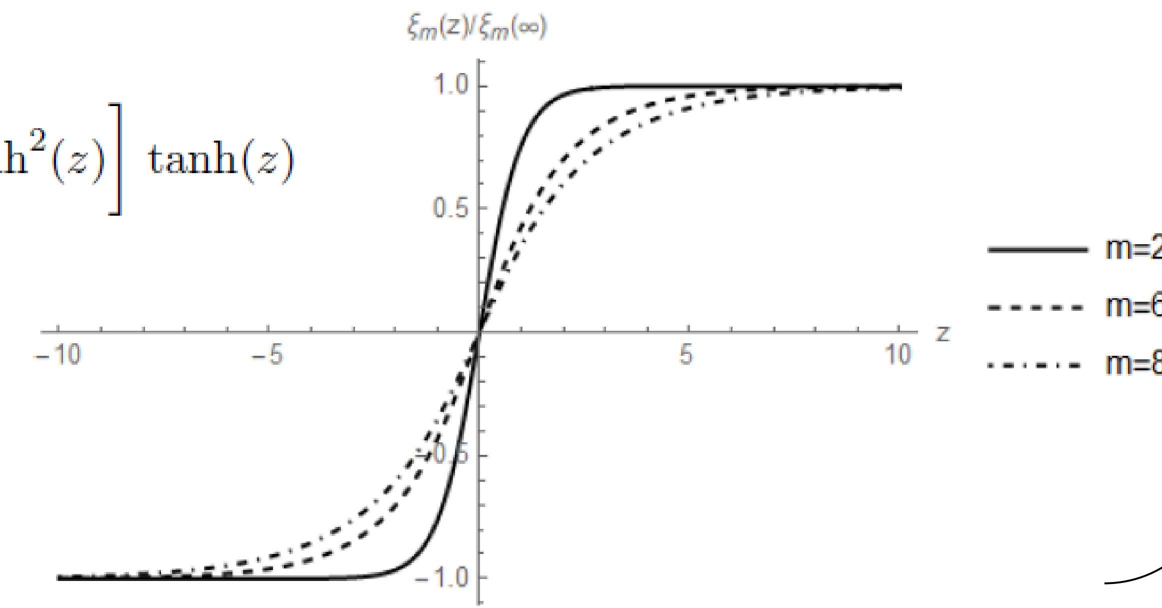
$$P = \frac{\kappa\,(-2X_\xi)^{m/2}}{2(m-1)} + \frac{1-\kappa}{2(n-1)}\left(1-\phi^2\right)^{2\left(1-\frac{n}{m}\right)}(-2X_\xi)^{n/2} + \left(1-\phi^2\right)^{2\left(1-\frac{2}{m}\right)} X_\xi$$

… Extends naturally on the whole real line for $\xi$

And should yield domain walls interpolating between non adjacent vacuua of the periodic function $\left(1-\phi^2\right)$, where $\phi = \phi(\xi)$ (cf. Sine-Gordon model)

NB: the domain wall profiles for the $\xi_p$ variable are

$$\xi(z) = {}_2F_1\left[\frac{1}{2}, 1-\frac{2}{m}, \frac{3}{2}, \tanh^2(z)\right]\tanh(z)$$



### III.2.2. Energy, Bogomolny and topology

Starting with the general form $$\mathcal{L} = \left(\sum_{n\in\mathbb{N}} \kappa_n(-2X_\psi)^{n/2}\right)\cosh^{-4}\psi.$$

### III.2.2. Energy, Bogomolny and topology

Starting with the general form $\mathcal{L} = \left( \sum_{n \in \mathbb{N}} \kappa_n (-2X_\psi)^{n/2} \right) \cosh^{-4} \psi$

The Hamiltonian density of an arbitrary field configuration reads

$$\mathcal{H}(t,z) = -\left( \sum_{n \in \mathbb{N}} \kappa_{2n} (\psi'^2 - \dot{\psi}^2)^n + 2n\kappa_{2n} \dot{\psi}^2 (\psi'^2 - \dot{\psi}^2)^{n-1} \right) \cosh^{-4} \psi$$

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e.g. for the mexican hat model we get $\mathcal{H}(t,z) = (1 + \psi'^2 + \dot\psi^2)/(2\cosh^4\psi)$

or, using the notation $x = \psi'$, $y = \dot\psi$

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The Bogomolny decomposition reads now simply

$$1 + x^2 + y^2 = y^2 + (x\pm 1)^2 \mp 2x$$

Kinetic energy

Vanishes for the wall configuration

Topological term

Yielding the boundary supported integral

$$\int \frac{\psi'}{cosh^4\,\psi}\,dz$$

Considering the same decomposition for $\mathcal{L} = \left( \sum_{n \in \mathbb{N}} \kappa_n (-2X_\psi)^{n/2} \right) \cosh^{-4} \psi$

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Defining $\Sigma_{\kappa,k} = \sum_{n\in\mathbb{N}} \kappa_{2n} n^k$

We get $\Pi_0(x,y) = (4\Sigma_{\kappa,1} - 2\Sigma_{\kappa,0}) \mp 4x\Sigma_{\kappa,1} + \Pi\left((x\mp 1), y^2\right)$

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Yielding the boundary
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$$\int \frac{\psi'}{cosh^4\,\psi}dz$$

Topological
term

$\Pi(a,b)$: polynomial in a and b
starting at quadratic order in a
and vanishing in (a=0,b=0)

The topological term $\mp 4x\Sigma_{\kappa,1}$ yields the same total energy and conserved charge as in the canonical model

$$\mathcal{H}_\infty(t) \;=\; \mp\Sigma_{\kappa,0}\int \frac{\psi'}{\cosh^4\psi}dz$$

Using here $\Sigma_{\kappa,0} = 2\Sigma_{\kappa,1}$

Associated to the same conserved current $\tilde{J}^\mu_\psi = \tilde{\mathcal{C}}\epsilon^{\mu\nu}\partial_\nu\left(\int_{\psi_0}^{\psi}\frac{du}{\cosh^4 u}\right)$

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The left over term $\Pi = \Pi_0 \pm 2x\Sigma_{\kappa,0}$

Yields the « non topological » part of the energy

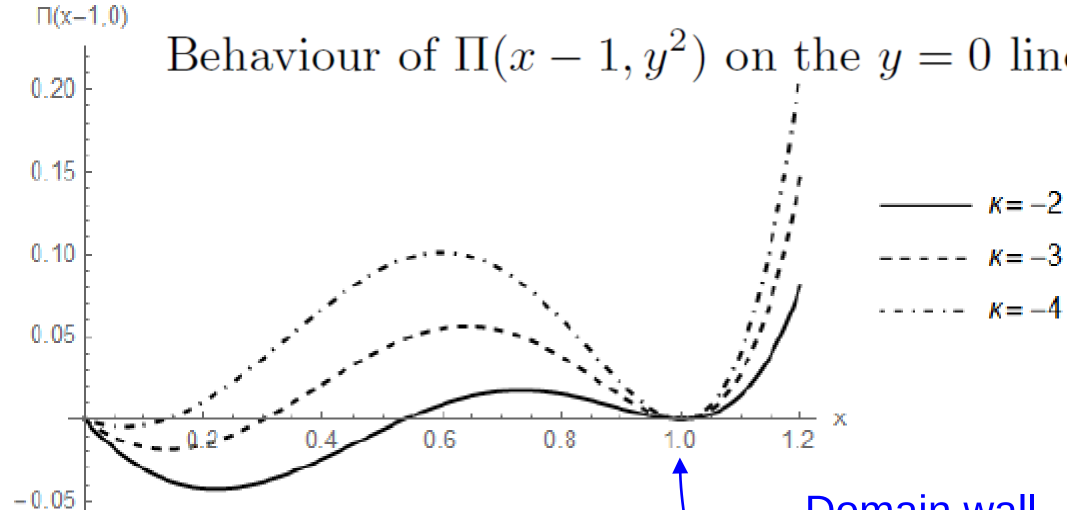
For non canonical domain walls models, while the total energy can be made everywhere positive, the non topological part of the energy can be shown to become somewhere negative.

E.g. for the model with Lagrangian

$$\frac{1}{\cosh^4\psi}\left(X_\psi+\frac{1-\kappa}{6}\left(-2X_\psi\right)^2+\frac{\kappa}{10}\left(-2X_\psi\right)^3\right)$$

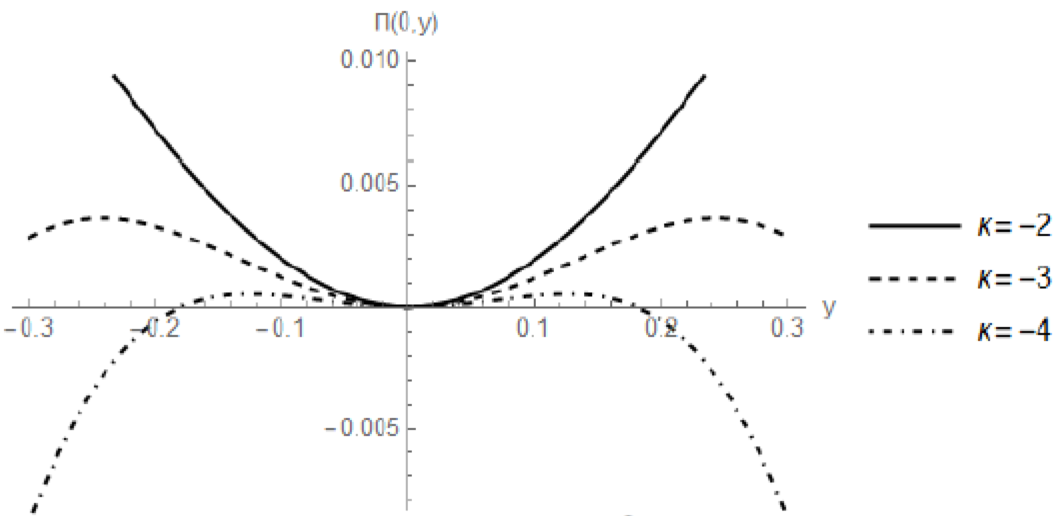
With $-5<\kappa<-1$

Behaviour of $\Pi(x-1,y^2)$ on the $y=0$ line




Domain wall solution: x=1, y=0

It is (only) a local minimum of the energy




Behaviour of $\Pi(x-1,y^2)$ on the $x=1$ line

### III.2.3. Wall perturbations

Writing $\phi(t,z) = \phi(z) + \varphi(t,z)$

The mode functions $\varphi = \sum \varphi_k(z) e^{i\omega_k t}$ obey at quadratic order

$$\left(\mathcal{Z}^{zz}\varphi_k'\right)' - \left(\mathcal{Z}^{00}\,\omega_k^2 + \mathcal{M}^2\right)\varphi_k = 0$$

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| | $P_{n,m}$ | $P_{4,6}$ | $P_{\rm can}$ |
|---|---|---|---|
| $\mathcal{H}(z)$ | $\frac{1}{2}\left(\frac{n-2}{n-1} + \frac{(m-n)\kappa}{(n-1)(m-1)}\right)\cosh^{-4}(z)$ | $\frac{5+\kappa}{15}\,\cosh^{-4}(z)$ | $\cosh^{-4}(z)$ |
| $\mathcal{Z}^{00}(z)$ | $-\frac{1}{2}\left(\frac{n-2}{n-1} + \frac{(m-n)\kappa}{(n-1)(m-1)}\right)$ | $-\frac{5+\kappa}{15}$ | $-1$ |
| $\mathcal{Z}^{zz}(z)$ | $\frac{2-n+(n-m)\kappa}{2}$ | $-\left(1+\kappa\right)$ | $1$ |
| $\mathcal{M}^2(\mathrm{z})$ | $(2-n+(n-m)\kappa)\left(3\phi^2(z)-1\right)$ | $-2\left(1+\kappa\right)\left(3\phi^2(z)-1\right)$ | $2\left(3\phi^2(z)-1\right)$ |

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The choice κ = -5/4 yields perturbations identical to the one of the canonical model « Mimicker model »

## Cubic vertices

$$\delta^{(3)}\mathcal{L} = -\frac{1}{3!}\left[\mathcal{Y}^{\mu\nu\rho}\,\partial_\mu\varphi\,\partial_\nu\varphi\,\partial_\rho\varphi - 3\mathcal{Y}^{\mu\nu}\,\varphi\,\partial_\mu\varphi\,\partial_\nu\varphi + \mathcal{Y}\,\varphi^3\right]$$

| | Generic $P_{n,m}$ | Mimicker $P_{n,m}$ | Mimicker $P_{4,6}$ |
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| $\mathcal{Y}^{00z}$ | $-\frac{3}{2}\left(\frac{n(n-2)(\kappa-1)}{n-1} - \frac{m(m-2)\kappa}{m-1}\right)\frac{1}{1-\phi^2}$ | $0$ | $0$ |
| $\mathcal{Y}^{zzz}$ | $\frac{n(n-2)(\kappa-1)-m(m-2)\kappa}{2}\,\frac{1}{1-\phi^2}$ | $\frac{mn(n-2)(m-2)}{2(2-n+m(n-1))}\,\frac{1}{1-\phi^2}$ | $\frac{6}{1-\phi^2}$ |
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| $\mathcal{Y}^{zz}$ | $-\left(n(n-2)(\kappa-1) - m(m-2)\kappa\right)\frac{\phi}{1-\phi^2}$ | $-\frac{mn(n-2)(m-2)}{(2-n+m(n-1))}\,\frac{\phi}{1-\phi^2}$ | $-\frac{12\phi}{1-\phi^2}$ |

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Strong coupling off the wall ☹

# III. Domain walls without a potential

## III.3. Issues, perspectives and conclusions

## This can easily be generalized in various ways

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e.g. moving walls (as in the canonical theory)

$$\phi_m(t,x) = \pm \tanh\left(\lambda \frac{z \pm \beta t}{\sqrt{1-\beta^2}}\right)$$

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Sine-Gordon like and other wall

From e.g. $\mathcal{L} = \left(\sum_{n\in\mathbb{N}} \kappa_n (-2X_\psi)^{n/2}\right)\cosh^{-2k}\psi$

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k=1 yields Sine-Gordon like walls (breather ?)

- With more than three non vanishing $\kappa_n$ above, several (locally stable) walls could coexist.

# Some problematic aspects

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- Strong coupling off the wall
- Solutions are only locally stable and theories locally sound

$$0 < P_X$$
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## Phenomenology and related issues

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## Phenomenology and related issues

- Wall decay ?

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Can this be changed in different theories
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**Phenomenology and related issues**

- Wall decay ?
- Early universe ?

**Thank you for your attention !**