Black hole perturbations in modified gravity

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Astroparticules
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Introduction

- **GW astronomy** provides new windows to **test GR,** in particular in the strong field regime.
- **Ringdown phase** of a BH merger is interesting for modified gravity models; it can be described by BH linear perturbations.
- Consider the most genera**l** framework of **scalar-tensor theories** with a single scalar degree of freedom: **DHOST** theories
- **Perturbations of BH** in DHOST theories studied in a few papers
- **Quasi-normal modes**: Schroedinger-like method and **alternative** approach

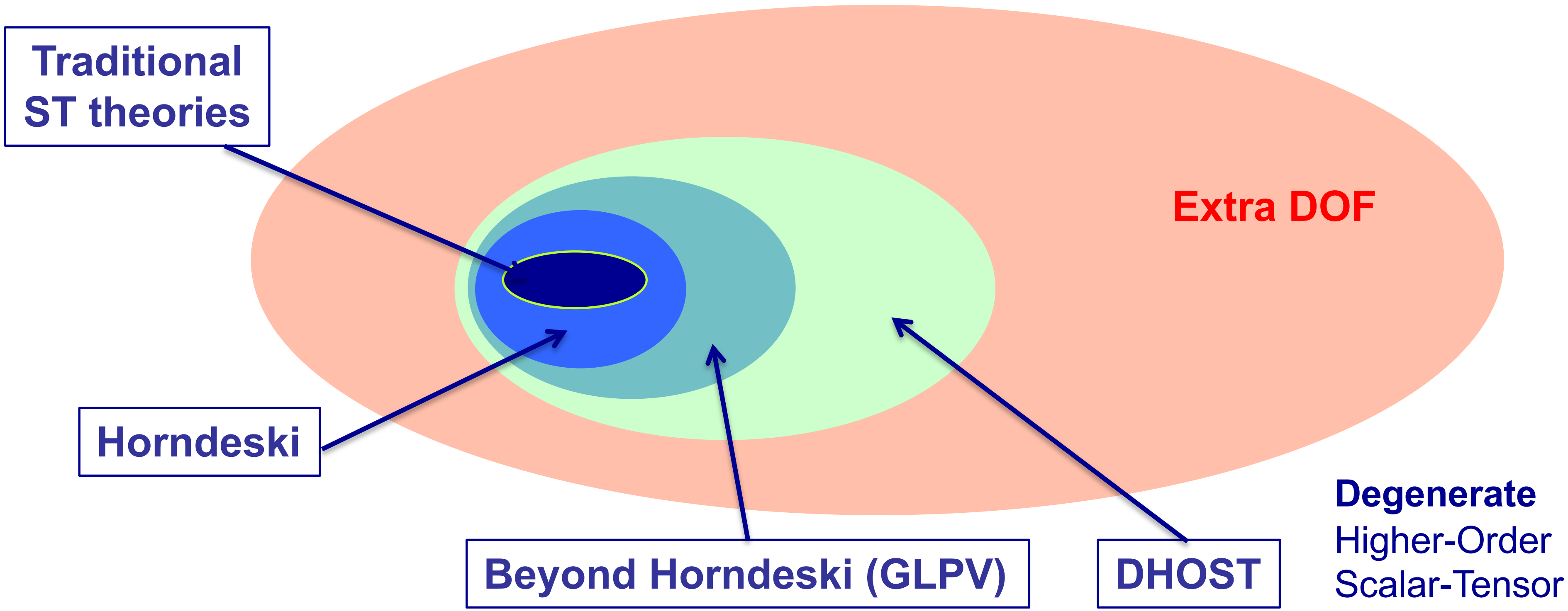
Based on DL, Noui & Roussille 2103.14744 & 2103.14750

DHOST theories

- Traditional theories: $\mathcal{L}(\nabla_\lambda \phi, \phi)$

- Generalized theories: $\mathcal{L}(\nabla_\mu \nabla_\nu \phi,\ \nabla_\lambda \phi,\ \phi)$

  DHOST: most general family of covariant scalar-tensor theories with a **single scalar DOF**



DHOST theories

- Action of **quadratic DHOST** [DL & Noui '15]

$$S=\int d^4x\sqrt{-g}\left[P(X,\phi)+Q(X,\phi)\,\Box\phi+F(X,\phi)\,R+\sum_{i=1}^{5}A_i(X,\phi)\,L_i^{(2)}\right]$$

$$L_1^{(2)}=\phi_{\mu\nu}\,\phi^{\mu\nu}\,,\quad L_2^{(2)}=(\Box\phi)^2\,,\quad L_3^{(2)}=(\Box\phi)\phi^{\mu}\phi_{\mu\nu}\phi^{\nu}$$
$$L_4^{(2)}=\phi^{\mu}\phi_{\mu\rho}\phi^{\rho\nu}\phi_{\nu}\,,\quad L_5^{(2)}=(\phi^{\mu}\phi_{\mu\nu}\phi^{\nu})^2$$

$$X\equiv\nabla_{\mu}\phi\nabla^{\mu}\phi$$
$$\phi_{\mu\nu}\equiv\nabla_{\nu}\nabla_{\mu}\phi$$

The functions F and A_I satisfy three **degeneracy conditions**.

- Extension to cubic order (in $\phi_{\mu\nu}$)

[Ben Achour, Crisostomi, Koyama, DL, Noui & Tasinato '16]

- Quadratic Horndeski

$$A_1=-A_2=2F_X\,,\quad A_3=A_4=A_5=0$$

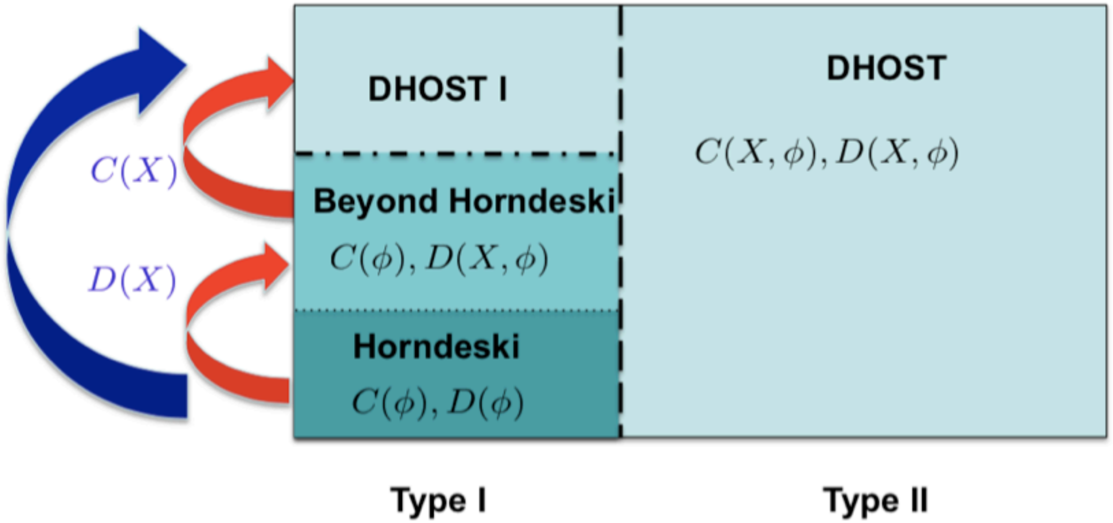
Disformal transformations

- Transformation $g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = C(X,\phi)\, g_{\mu\nu} + D(X,\phi)\, \partial_\mu\phi\, \partial_\nu\phi$

- From an action $\tilde{S}\,[\phi, \tilde{g}_{\mu\nu}]$, one gets the new action

$$S[\phi, g_{\mu\nu}] \equiv \tilde{S}\,[\phi, \tilde{g}_{\mu\nu} = C\, g_{\mu\nu} + D\, \phi_\mu \phi_\nu]$$

- DHOST family is **complete** under these transformations



- When **standard fields** are (minimally) included, two disformally related theories are **physically inequivalent** !

Black hole solutions

- One can find BH solutions with a **nontrivial** scalar field
  - Metric: $$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$
  - Scalar field: $$\phi(t,r) = q\,t + \chi(r)\,,$$ [Babichev & Charmousis '13]

  [$q \neq 0$ possible in shift-symmetric theories]

- **Stealth Schwarzschild:** $A(r) = 1 - \dfrac{r_s}{r}$ $\qquad X(r) = -q^2$

$$F(X) = 1 + \alpha(X+q^2) + \beta(X+q^2)^2/2$$

- **Non-stealth (BCL):** [Babichev, Charmousis & Lehébel '17]

$$A(r) = \left(1 - \frac{r_+}{r}\right)\left(1 + \frac{r_-}{r}\right), \quad q = 0 \qquad X(r) \propto \frac{1}{r^4}$$

$$F(X) = f_0 + f_1\sqrt{X}, \quad P(X) = -p_1 X$$

Black hole perturbations

- Time Fourier transform:

$$f(t,r) = \int d\omega\, f(\omega,r) e^{-i\omega t}$$

- **Axial** (or odd) modes: $h_0(r), h_1(r)$

$$h_{\mu\nu} = \sum_{\ell,m} \begin{pmatrix} 0 & 0 & \frac{1}{\sin\theta} h_0^{\ell m} \partial_\varphi & -\sin\theta h_0^{\ell m} \partial_\theta \\ 0 & 0 & \frac{1}{\sin\theta} h_1^{\ell m} \partial_\varphi & -\sin\theta h_1^{\ell m} \partial_\theta \\ \text{sym} & \text{sym} & 0 & 0 \\ \text{sym} & \text{sym} & 0 & 0 \end{pmatrix} Y_{\ell m}(\theta,\varphi)$$

- **Polar** (or even) modes: H_0, H_1, H_2, K (and $\delta\phi$)

$$h_{\mu\nu} = \sum_{\ell,m} \begin{pmatrix} A(r) H_0^{\ell m}(r) & H_1^{\ell m}(r) & 0 & 0 \\ H_1^{\ell m}(r) & A^{-1}(r) H_2^{\ell m}(r) & 0 & 0 \\ 0 & 0 & K^{\ell m}(r)\, r^2 & 0 \\ 0 & 0 & 0 & K^{\ell m}(r)\, r^2 \sin^2\theta \end{pmatrix} Y_{\ell m}(\theta,\varphi)$$

Axial modes in GR

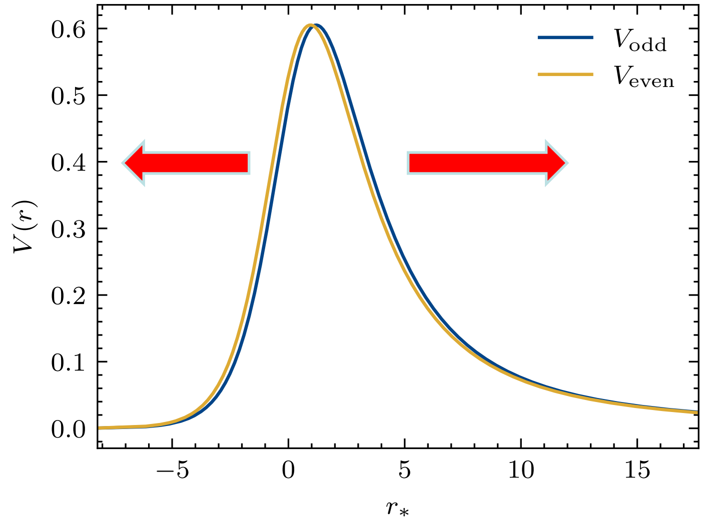
- The linearised metric eqs yield only 2 independent eqs

$$\frac{dY}{dr} = M(r)\,Y(r)\,, \qquad Y = \begin{pmatrix} h_0 \\ h_1/\omega \end{pmatrix}$$

or, in a Schroedinger form, [Regge & Wheeler '57]

$$\frac{d^2\hat{Y}}{dr_*^2} + \left(\omega^2 - V(r)\right)\hat{Y} = 0$$

[r_* tortoise coordinate]



- Asymptotically ($r_* \to -\infty, +\infty$)

$$e^{-i\omega t}\hat{Y}(r) \approx \mathcal{A}\, e^{-i\omega(t-r_*)} + \mathcal{B}\, e^{-i\omega(t+r_*)}$$

outgoing ingoing

- **Quasi-normal modes**: $\mathcal{A}_{\mathrm{hor}} = 0 \quad \text{and} \quad \mathcal{B}_{\infty} = 0$

$$\omega_n = \omega_{R\,n} + i\,\omega_{I\,n} \qquad (\omega_{I\,n} < 0)$$

Axial modes in DHOST

- The equations have a similar structure:

$$\frac{dY}{dr} = MY\,, \quad M \equiv \begin{pmatrix} 2/r + i\omega\Psi & -i\omega^2 + 2i\lambda\Phi/r^2 \\ -i\Gamma & \Delta + i\omega\Psi \end{pmatrix}$$

which can be rewritten, via $Y(r) = \hat{P}(r)\,\hat{Y}(r)$, as

$$\frac{d\hat{Y}}{dr_*} = \begin{pmatrix} i\omega\,n\Psi & 1 \\ V(r) - \omega^2/c^2(r) & i\omega\,n\Psi \end{pmatrix} \hat{Y} \qquad \frac{dr}{dr_*} \equiv n(r)$$

- After the time redefinition $t \longrightarrow t - \int \Psi(r)dr$, one gets

$$\frac{d^2\hat{Y}_1}{dr_*^2} + \left(\frac{\omega^2}{c^2(r)} - V(r)\right)\hat{Y}_1 = 0$$

where *V(r)* and *c(r)* depend on the choice *n(r).*

Axial modes in stealth Schwarzschild

- The matrix depends on

$$\Psi=\frac{\zeta\, r_s^{1/2} r^{3/2}}{(r-r_s)(r-r_g)}\,,\qquad \Phi=\frac{r-r_g}{(1+\zeta)r}\,,\qquad \Gamma=\frac{(1+\zeta)r^2}{(r-r_g)^2}\,,\qquad \Delta=\frac{1}{r}-\frac{1}{r-r_g}$$

where $\zeta\equiv 2q^2\alpha\,,\qquad r_g\equiv(1+\zeta)r_s\qquad [\,\zeta=0:\ \mathrm{GR}\,]$

- If r_* is the tortoise coordinate then $c(r)=\dfrac{r-r_g}{\sqrt{1+\zeta}\,(r-r_s)}$

- If $r_*=\sqrt{1+\zeta}\,[\,r+r_g\ln(r/r_g-1)]$ so that $c(r)=1$, one finds

$$V_{c=1}(r)=\left(1-\frac{r_g}{r}\right)\frac{2(\lambda+1)r-3r_g}{(1+\zeta)\,r^3}$$

[see also Tomikawa & Kobayashi '21]

This can be understood via a **disformal transformation** to the « frame » where $A_1=0$.

Axial modes in stealth Schwarzschild

Using the disformal transformation

$$\tilde{g}_{\mu\nu} = C g_{\mu\nu} + D\, \partial_\mu \phi\, \partial_\nu \phi\,, \qquad \frac{D}{C} = \frac{2\alpha}{1+\zeta}\,.$$

one gets the metric

$$d\tilde{s}^2 = \frac{C}{1+\zeta}\left[-\left(1-\frac{r_g}{r}\right)dt^2 + 2\zeta\frac{\sqrt{r_s r}}{r-r_s}dr\,dt + \frac{(1+\zeta)r^2 - r r_s}{(r-r_s)^2}dr^2 + (1+\zeta)r^2 d\Omega^2\right]$$

or, with $d\tilde{t} \equiv dt - \Psi dr$,

$$d\tilde{s}^2 = \frac{C}{1+\zeta}\left[-\left(1-\frac{r_g}{r}\right)d\tilde{t}^2 + \frac{1+\zeta}{1-\frac{r_g}{r}}dr^2 + (1+\zeta)r^2 d\Omega^2\right]$$

which is Schwarzschild (up to coordinate rescalings).

Asymptotics of a diffential system

- Instead of a Schroedinger-like approach, one can use directly the initial first-order equations of motion and their asymptotic limit:

$$\frac{dY}{dz} = M(z)\,Y\,, \quad M(z) = M_r z^r + M_{r-1} z^{r-1} + \ldots \quad (z \to \infty)$$

- The generic solution is of the form [Wasow '65]

$$Y(z) = e^{\boldsymbol{\Upsilon}(z)}\, z^{\boldsymbol{\Delta}}\, \mathbf{F}(z)\, Y_0, \qquad (z \to \infty)$$

- There exists a well-defined algorithm to determine the diagonal matrices $\boldsymbol{\Upsilon}(z)$ and $\boldsymbol{\Delta}$. [Balser '99]

Idea: diagonalise, order by order, the matrix *M*, with $Y(z) = P(z)\,\tilde{Y}(z)$

$$\frac{d\tilde{Y}}{dz} = \tilde{M}(z)\,\tilde{Y}, \qquad \tilde{M}(z) \equiv P^{-1} M P - P^{-1}\frac{dP}{dz}$$

Axial modes of BCL black hole

$$A(r) = \left(1 - \frac{r_+}{r}\right)\left(1 + \frac{r_-}{r}\right)\,, \quad r_m \equiv r_+ - r_-$$

- **At spatial infinity:**

$$M(r) = M_0 + \frac{1}{r}M_{-1} + \ldots\,, \quad M_0 \equiv -i\begin{pmatrix} 0 & \omega^2 \\ 1 & 0 \end{pmatrix},\; M_{-1} \equiv 2\begin{pmatrix} 1 & 0 \\ -ir_m & 0 \end{pmatrix}$$

$$\frac{d\tilde{Y}}{dr} = \tilde{M}\tilde{Y}\,, \quad \tilde{M}(r) = \begin{pmatrix} -i\omega & 0 \\ 0 & i\omega \end{pmatrix} + \frac{1}{r}\begin{pmatrix} 1 - i\omega r_m & 0 \\ 0 & 1 + i\omega r_m \end{pmatrix} + \ldots$$

Solution

$$\tilde{Y}(r) \;\approx\; \begin{pmatrix} a_-\, e^{-i\omega r} r^{1-i\omega r_m} \\ a_+\, e^{+i\omega r} r^{1+i\omega r_m} \end{pmatrix} \approx r \begin{pmatrix} a_-\, e^{-i\omega r_*} \\ a_+\, e^{+i\omega r_*} \end{pmatrix}$$

where $r_* = \int dr/A(r)$ (tortoise coordinate)

- **Near the horizon**: expansion in $\varepsilon \equiv r - r_+$ yields the asymptotic solution

$$e^{-i\omega t}\tilde{Y}_\mp \approx a_\mp e^{-i\omega(t \pm \eta r_*)} \qquad\qquad c(r_+) = \eta^{-1}$$

Axial modes of BCL black hole

- **Computation of QNMs** using a spectral method

Ansatz: $$Y = e^{i\omega r} r^{1+i\omega r_m} \left(1 - r_+/r\right)^{-i\omega r_0} \begin{pmatrix} f_1(r) \\ \left(1 - r_+/r\right)^{-1} f_2(r) \end{pmatrix}$$

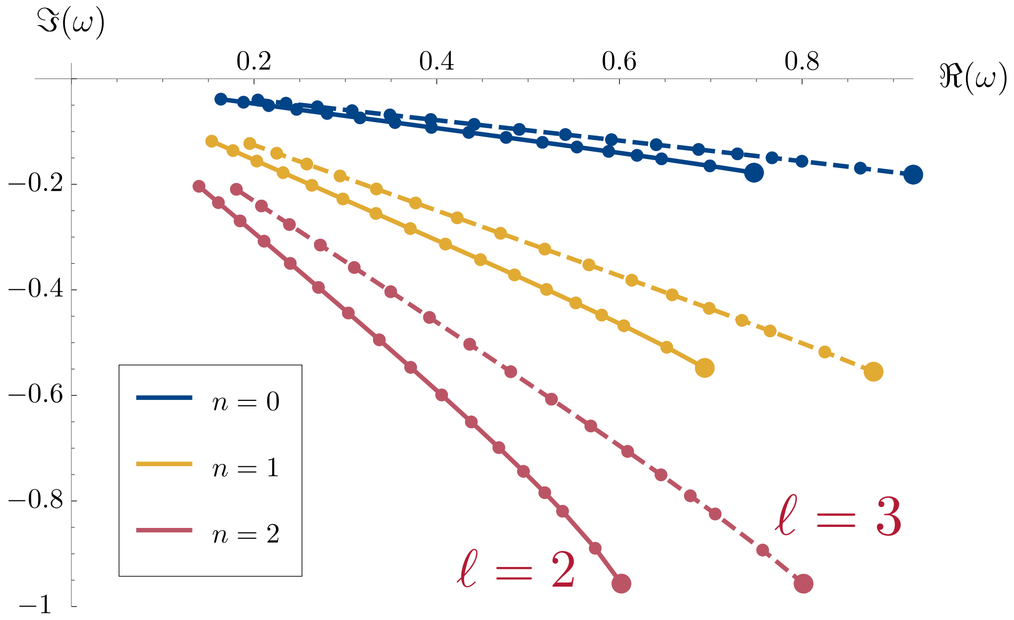
Decomposing in terms of Chebyshev polynomials,

$$f_i(u) = \sum_n \alpha_{i,n} T_n(u)$$

$$u = 2\frac{r_+}{r} - 1 \in [-1, +1]$$

one gets an algebraic linear system

$$M_N(\omega)\, V_N(\alpha_{i,n}) = 0$$



Polar modes

- The linearised metric equations yield
  - 2 independent equations in GR (1 dof)
  - **4 independent equations** in DHOST theories (2 dof)

- In GR, one gets a 2-dimensional system $Y' = M\,Y$, which can be written in a Schroedinger form. [Zerilli '70]

- In DHOST theories, the system $Y' = M\,Y$ is now 4-dimensional, with

$$Y = {}^T(K \;\; \delta\phi \;\; H_1 \;\; H_0)$$

- If there is no generalised Schroedinger-like system, one can still use the **asymptotic approach.**

Polar modes of BCL black holes

- One can study the **asymptotic behaviour** of the 4-dim system at spatial infinity and near the horizon, and **extract the independent modes**.

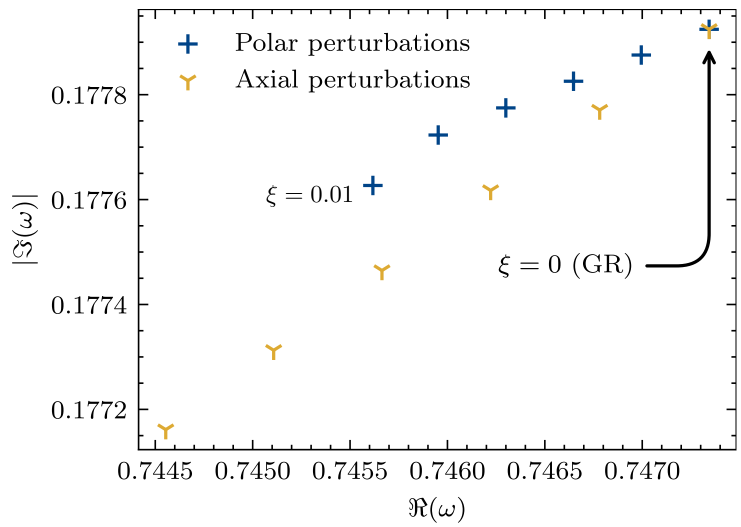
- At spatial infinity, one gets

$$\tilde{Y}(r) \approx \begin{pmatrix} c_- \, r^{-i\omega r_m} \, e^{-i\omega r} \\ c_+ \, r^{+i\omega r_m} \, e^{+i\omega r} \\ \frac{d_-}{r^3} \, r^{-\frac{\omega r_m}{\sqrt{2}}} \, e^{-\sqrt{2}\omega r} \\ \frac{d_+}{r^3} \, r^{+\frac{\omega r_m}{\sqrt{2}}} \, e^{+\sqrt{2}\omega r} \end{pmatrix}$$

2 « gravitational » modes,
2 « scalar» modes

- Similar results near the horizon

- « Gravitational » quasi-normal modes



Conclusions

- Analysis of the **BH linear perturbations** in **DHOST** theories
- **Axial modes**: one can recover a **Schroedinger-like equation** with some effective potential and propagation speed (which depend on the choice of radial coordinate).
- In **stealth Schwarzschild**, the gravitational modes **“see”** a **different Schwarzschild metric**, corresponding to the DHOST “frame” where the speed of GWs is constant.
- **Polar modes:** the structure is much more complicated than in GR (4-dim system). One finds “gravitational” and “scalar” modes.
- **Systematic approach** to **decouple the modes asymptotically**.
- Future work: other solutions; ultimately rotating blackholes…