

Quantization of Massless Particles and Gauged Poisson Sigma Models

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1. 1st quantization

- Symplectic geometrodynamics
and Vasiliev's higher-spin gravity
- W strings

2. 2nd quantization

- AKS τ iomatic higher-spin gravity
- Symplectic structure for
massless particles

3. 3rd quantization

- Fractional-spin gravity
as a holographic defect
a la Nilsson-Vasiliev

Newtonian particle

$$\mathcal{S} = \int d\tau \left(\frac{\dot{q}^2}{t} - tV \right)$$

$$\approx \int \left(\theta_s + s dt + a(H_s + s) \right)$$

Symplectic mfd S : $\omega_s = d\theta_s$

Evolution group \mathbb{R} : $\omega_{T^*\mathbb{R}} = ds \wedge dt$

Total Hamiltonian $\mu = H_s + s \approx 0$

$\omega_s, H_s \ni$ background fields on S

States $B : S \times \mathbb{R} \rightarrow \mathbb{R}$

Relativistic version

Evolution G : $H_a : S \rightarrow \mathbb{R}$

Mass-shells K : $K_r : S \times G \rightarrow \mathbb{R}$

$$\mathcal{S} = \int \left(\theta_s + s_m dt^m + a^m (s_m + \Omega_m^a H_a) + \lambda^r K_r \right)$$

Maurer-Cartan $\Omega := dt^m \Omega_m^a H_a$

Background EoM

$$d_G \Omega + \frac{1}{2} \{ \Omega, \Omega \}_s = 0$$

$$d_G K_r + \{ \Omega, K_r \}_s = 0$$

$$\{ K_r, K_s \}_s = f_{rs}^t K_t$$

State EoM

$$d_G B + \{ \Omega, B \}_s = 0$$

$$\{ K_r, B \}_s = 0$$

$$B \sim B + \varepsilon^r K_r$$

Consider

$$K = \mathfrak{gl}(1) \times K'$$

Dynamical BRST operator

$$Q = \underset{\substack{\uparrow \\ C^m S_m}}{g_{T^*G}} + X \in \mathfrak{gh}_{\mathfrak{gl}(1)} = \begin{bmatrix} e & f \\ \tilde{f} & \tilde{e} \end{bmatrix}$$

↓ deg

Quillen superconnection

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$X \in H^0(g_{T^*G}) \cong \Omega_{\text{hor}}(T^*G)$$

Frobenius -
Chern-Simons

$$d_G X + X * X \cong 0$$

Decompose $X = Ae + \tilde{A}\tilde{e} + Bf + \tilde{B}\tilde{f}$

$$A, \tilde{A}, B, \tilde{B} \in \Omega(G) \otimes \mathcal{U}(S) \otimes \mathfrak{gh}_{K'}$$

↑ operator algebra

$$d_G A + A * A + B * \tilde{B} \cong 0$$

$$d_G \tilde{A} + \tilde{A} * \tilde{A} + \tilde{B} * B \cong 0$$

$$d_G B + A * B - B * \tilde{A} \cong 0$$

$$d_G \tilde{B} + \tilde{A} * \tilde{B} - \tilde{B} * A \cong 0$$

← dynamical Hamiltonian and BRST operator

← nonlinear evolution equation of state

← defects on G
↪ back-reaction

Background

$$\langle X \rangle = \Omega(e + \tilde{e}) + Q_{K'} + I_G$$

Deform $\Theta_S : \Omega(S) \xrightarrow{\cong f_{1,2}} \Omega(M) / \Omega_{\text{hor}}(M)$

M differential Poisson mfd

$$\{f_1, f_2\} = \pi^{\alpha\beta} \nabla(\Gamma)_\alpha f_1 \wedge \nabla(\Gamma)_\beta f_2 + (\pi R(\Gamma))^{\alpha\beta} i_\alpha f_1 \wedge i_\beta f_2$$

with special fundamental vertical vector fields generating S -fibration.

Tool: $G \times K$ gauged AKSZ sigma model on $T[1]M \times T^*G$ (Zucchini)

$$\begin{aligned} \mathcal{S} = & \int \int \left(\eta_\alpha \wedge (\mathbb{D}\phi^\alpha + \Theta^\alpha) + \frac{1}{2} \eta_\alpha \wedge \eta_\beta \pi^{\alpha\beta} \right. \\ & + \chi_\alpha \wedge \nabla_{\mathbb{D}\phi}(\tilde{\Gamma}) \Theta^\alpha + \frac{1}{4} \chi_\alpha \wedge \chi_\beta (\pi R(\tilde{\Gamma}))^{\alpha\beta} \chi_\gamma \Theta^\gamma \Theta^\delta \\ & + \mathcal{J}_m \wedge \mathbb{D}t^m + \mathcal{J}^m \wedge \mathbb{D}S_m + \mathcal{J}_m \wedge \mathcal{J}^m \\ & + \sum_a \alpha \wedge F^a + \beta^a \wedge (\mathbb{D}\gamma_a + \mu_a + \sum_a) \\ & \left. + \sum_r \alpha \wedge F^r + \beta^r \wedge (\mathbb{D}\gamma_r + K_r + \sum_r) \right) \end{aligned}$$

where

$$\mathbb{D}\phi^\alpha = d\phi^\alpha + a^\alpha \pi^{\alpha\beta} \partial_\beta \mu_a + \lambda^r \pi^{\alpha\beta} \partial_\beta K_r$$

$$\mu_a = K_a^m S_m + H_a \approx 0 \text{ at bdry}$$

Two types of supersymmetries:

BRST transformations

$$s \phi^\alpha = \theta^\alpha, \quad s \theta^\alpha = 0, \dots$$

\leadsto de Rham differential on M
contained in BV-BRST differential

Fermionic vertical shift symmetries

$$\delta \phi^\alpha = 0, \quad \delta \theta^\alpha = V^\alpha, \dots$$

\leadsto closed and central fiber
volume form $\underline{\mathbb{1}} \in \Omega(M)$

Differential graded associative algebra

$$(\Omega(M), d, *)$$

with supertrace operation \int_M and horizontal
differential subalgebra

$$\Omega_{\text{hor}}(M) := \frac{\Omega(M)}{\ker \underline{\mathbb{1}}^*} \cong \{[f] \mid f \sim f + t, \underline{\mathbb{1}}^* t = 0\}$$

$$d[f] = [df], \quad [f] * [g] = [f * g]$$

$$\int_M [f] = \int_M \underline{\mathbb{1}}^* f$$

Dynamical BRST operator (symplectic case)

$$Q = q_{T^*G} + q_M + X =: X$$

$$q_M = \underbrace{s_M(e + \tilde{e})}_{\theta^x \phi_x} + \underbrace{f - \omega_M f}_{S_M \times S_M}$$

Superconnection

large symplectomorphisms

$$X \in \Omega_{hor}(G \times M) \otimes \mathfrak{g} \otimes \mathbb{C}[\Gamma]$$

$$dX + X * X + [\omega_M](e + \tilde{e}) = 0$$

does not affect Vasiliev branch - disregard!

Background

$$\langle X \rangle = (\Omega + Q_{K'}) / (e + \tilde{e}) + [I_{G \times M}]$$

defects on $T^*G \times M$

$I_{G \times M}$ = 2-forms with $g_{K'} = 0$ \sim conical singularities of co-dimension 2

\rightarrow B backreacts to symplectic geometry on M and evolution Hamiltonians on G

Conformal particle

→ Vasiliev's Type A model

Noncommutative geometry

$$G = Sp(2n)$$

↑ spacetime

$$K' = sp(2)$$

$$S = \mathbb{R}^{2n} \hookrightarrow M = \mathbb{R}^{4n} \longrightarrow \mathbb{R}^{2n} \cup \begin{cases} \text{points} \\ \text{at} \\ \infty \end{cases}$$

↑
Y-space
(unbroken susy)

↑
Z-space
(broken susy)

Background for X

$$\Omega \in Sp(2n) \xrightarrow{Q_{sp(2)}} \Omega \in so(p, q)$$

$p+q=n$

$$\mathcal{U}(Y) = \bigoplus Sp(2n) \text{ tensors} \xrightarrow{Q_{sp(2)}} \bigoplus so(p, q) \text{ tensors}$$

$$\xrightarrow{I_M} \bigoplus so(p-1, q) \text{ tensors}$$

Vasiliev branch

$$X = \begin{bmatrix} W & B \\ I_M & W \end{bmatrix} \rightsquigarrow \begin{aligned} dW + W * W + B * I_M &\approx 0 \\ dB + W * B - B * W &\approx 0 \end{aligned}$$

$\mathcal{G}^h_{sp(2)} \Rightarrow$ Cartan integrability
beyond linear level

Bosonic W-string in AdS

→ stresses $gl(1)$ and \mathbb{Z}

Extend conformal particle on S^1 to

chiral symplectic boson with $h = \frac{1}{2}$

$$Y : T^2 \rightarrow \mathbb{R}^{2n}$$

$$c_{Vir} = -n, \quad k_{sp(2)} \neq 0$$

Spectral flow $P \in \mathbb{Z}$ units

→ 0-modes $Y_n^{(0)}$, $n \in \left\{ -\frac{|P|-1}{2}, \dots, \frac{|P|-1}{2} \right\}$

→ P copies of S

Gauge

$$W_{sp(2)+\infty} = \begin{matrix} sp(2) \\ h=1 \end{matrix} \oplus \begin{matrix} Vir \\ h=2 \end{matrix} \oplus \begin{matrix} \mathbb{Z} \\ h=3 \end{matrix} \oplus \begin{matrix} \mathbb{Z} \\ h=4 \end{matrix} \oplus \dots$$

$$c_{Vir}^{gh} = 2, \quad k_{sp(2)}^{gh} = 0$$

Idea: Add an anomalous sector

- restore nilpotency of Q .
- preserve target symmetries
- factored out on $H^0(Q)$

Proposal: Additional

matter (Z, θ, ϱ^*) with $c = n$

ghost: $\underset{h=0}{\sim} \quad$ with $c = -2$

$$\leadsto C_{\text{vir}}^{\text{tot}} = 0, \quad k_{\text{sp}(2)}^{\text{tot}} = 0$$

$P = 1$ sector:

$$W_{\text{Vir} + \text{cc}} \leadsto \mathcal{U}(Y_n^{\text{col}})$$

$$\mathfrak{gl}(1) \oplus \mathfrak{sp}(2) \leadsto \langle Q \rangle = \underset{\uparrow}{\mathfrak{q}_2} + \mathcal{Q}_{\text{sp}(2)}$$

$s_2(e + \tilde{e}) + f - \omega_2 \tilde{f}$

Dynamical BRST

$$Q = s_2(e + \tilde{e}) + X, \quad X \in H^1(\mathfrak{q}_2) \simeq \mathcal{U}(Z)$$

$$d_2 X + X * X + \omega_2(e + \tilde{e}) = 0$$

$O_{Sp}(W|4)$ W -string

$$\left(\underbrace{y, \bar{y}, \xi}_{\substack{2 \quad \bar{2} \quad W}}; \underbrace{z, \bar{z}, \theta, \bar{\theta}, \theta, \bar{\theta}^*}_{\substack{2 \quad \bar{2} \quad 2 \quad \bar{2} \quad 2 \quad \bar{2}}} \right)$$

matter anomalous

$$C_{vir} = -2 + \frac{W}{2}$$

$$C_{vir} = 2$$

Gauge

$$W_{0+\infty} = \underbrace{gl(1)}_{h=0} \oplus \underbrace{Vir}_{h=2} \oplus \underbrace{\dots}_{h=4} \oplus \dots$$

$$\Rightarrow C_{Vir}^{gh} = C_0^{gh} + C_{2+\infty}^{gh} = -2 - 1$$

$$\Rightarrow W = 6$$

Bosonic truncation to y, \bar{y} (Engqvist)

\Rightarrow affine $Sp(4) \cong$ affine $so(2,3)$
with critical level

Integrable spectral flow

$$\Rightarrow \left(\text{Di-Rac sign}(P)\text{-singletons} \right)^{\otimes |P|}$$

Classical solution spaces

Projection $\Sigma \times M \xleftarrow{P} G \times M$

Differential $\Omega_{\text{hor}}(\Sigma \times M) \xrightarrow{P^*} \Omega_{\text{hor}}(G \times M)$

graded subalgebra $W \xrightarrow{\quad} P^* X$

Classical moduli on Vasiliev branch

$W \rightsquigarrow$. BTZ-like flat connections
 . Transition functions

$I \rightsquigarrow$ Closed central 2-forms

$B \rightsquigarrow$ Integration constants $\in U(Y)$

Discrete duality groups

G

Particles
 Type D, N, ...
 Domainwalls
 Instantons
 ...

Subalgebras and intertwiners

• Boundary conditions on $\Sigma \times \mathbb{Z}$

• Strict operator algebras ($t_h = 1$)

\rightsquigarrow Non-commutative higher-spin geometries forming on-shell free differential subalgebras $\mathcal{F}[X]$ of $\Omega_{\text{hor}}(G \times M)$

2nd quantization

Non-commutative AKSZ sigma models
with star-product local BV algebras,

$$S_H = \int_{\hat{\Sigma} \times M} \underline{Y} * (P_\alpha \lrcorner dX^\alpha + H_*(X, P))$$

Attach operator algebras $\mathcal{F}[x]$ to defects

where AKSZ momenta vanish

$$\begin{array}{ccc} \hat{\Sigma} \times M & \xrightarrow{y} & T^* N \\ \downarrow & & \downarrow \\ G \times M \xrightarrow{p} \Sigma \times M & \xrightarrow{y} & N \end{array}$$

Two types of quantum corrections:

- Partial actions K attached to defects

\leadsto defect Ward identities and on-shell actions

- Extend $|X| = 1 \rightarrow |X| \in 2\mathbb{Z} + 1$

(Lyakhov
Sharapov
Skvortsov)

$$H = U + P_\alpha \lrcorner Q^\alpha + \sum \text{homotopy Poisson structures}$$

(Kontsevich)

Discrete symmetry

$$\begin{bmatrix} A & B \\ \check{B} & \check{A} \end{bmatrix} \xleftrightarrow{\mathbb{Z}_2} \begin{bmatrix} \check{A} & \check{B} \\ B & A \end{bmatrix}$$

Massless particle modes

$$B_{[0]} = \kappa \left(\Phi^{(+)} + \Phi^{(-)} \right) ; \left(\Phi^{(+)} \right)^\dagger = \Phi^{(-)}$$

$$\tilde{B}_{[0]} = \kappa \left(\tilde{\Phi}^{(+)} + \tilde{\Phi}^{(-)} \right) ; \left(\tilde{\Phi}^{(+)} \right)^\dagger = \tilde{\Phi}^{(-)}$$

positive / negative frequencies

Defect

$$\Sigma = S^1 \hookrightarrow \hat{\Sigma} = D_2$$

Partial action

$$K = \int_{\text{odd}} \text{Str} \left(\frac{1}{2} X \llcorner dX + \frac{1}{3} X \llcorner X \llcorner X \right) + \dots$$

presymplectic

$$\int \text{Str} \tilde{B} \llcorner dB$$

\rightsquigarrow Hyperkähler
 (holomorphic)
 symplectic

$\xrightarrow{\mathbb{Z}_2}$

Kähler
 symplectic

Hamiltonian zero-form constraints

$$(dX + X * X)|_{[0]} \equiv \begin{bmatrix} B_{[0]} * \tilde{B}_{[0]} & 0 \\ 0 & \tilde{B}_{[0]} * B_{[0]} \end{bmatrix} \approx 0$$

Relax by coupling to T^*g

$$\dots = - \int \text{Str } Y * g^{-1} Dg + \dots$$

Boundary equations of motion

$$\begin{cases} F + Y \approx 0, & F := dX + X * X \\ D Y \approx 0, & D Y := dY + X * Y - Y * X \\ D g \approx 0, & D g := dg - g * X \end{cases}$$

Classical anomaly $\vec{Q}^2 \sim g * Y \vec{D} g$

Three problems / remedies

BRST anomaly / cancels bulk anomaly U

$(Dg)|_{[0]} \approx 0$ / left-right gauging

$F|_{[0]} \approx 0$ / add odd section τ
 $\Rightarrow \cancel{\mathbb{Z}_2}$ $\Rightarrow \underline{\mathbb{Z}_2}$

BV master action

$$S_H = \int_{D_2} \text{Str} \left(Y^2 + \overset{\check{U}}{P}(F+Y) + P'(F' - \overset{\check{U}}{Y}') \right) \\ + \frac{1}{2} P^2 + \frac{1}{2} P'^2 \\ + \int g^{-1} Dg + \Lambda DY + \int \Lambda - \Lambda^2 Y \\ \uparrow dg + X'g - gX \quad \uparrow \text{KKS}$$

$$K = \int_{S^1} \text{Str} \left(\frac{1}{2} X dX + \frac{1}{3} X^3 + \frac{1}{2} X' dX' + \frac{1}{3} X'^3 \right. \\ \left. - Y g^{-1} Dg \right)$$

BV master equation

$$(S - \frac{1}{2} K, S - \frac{1}{2} K) \stackrel{\text{BV}}{=} 0$$

holds by virtue of

$$\{H, H\} \equiv 0 \Leftrightarrow \begin{cases} \vec{Q} U \equiv 0 \\ \vec{Q}^2 + \vec{\pi} dU \equiv 0 \\ [\vec{Q}, \vec{\pi}] \equiv 0 \\ [\vec{\pi}, \vec{\pi}] \equiv 0 \end{cases}$$

∴ Cartan integrable equations of motion that reduce to boundary system with $Y \approx 0$

⇒ Classical Vasiliev branch (~~\mathbb{H}_2~~)

Zero-form constraints

$$B_{[0]} * \tilde{B}_{[0]} + Y_{[0]} \approx 0 \quad \leftarrow F \approx 0$$

$$B'_{[0]} * \tilde{B}'_{[0]} + Y'_{[0]} \approx 0 \quad \leftarrow F' \approx 0$$

$$\left((B')^g \right)_{[0]} \approx B_{[0]} \quad \leftarrow Dg \approx 0$$

Semi-classically $sG \approx 0$

$$\rightsquigarrow d_g G + \left\{ \Omega_g, G \right\}_{\pi} \approx 0$$

$$\rightsquigarrow G(g, B_{[0]}, \tilde{B}_{[0]}) = g_{[0]}^{-1} * G(1, B_{[0]}, \tilde{B}_{[0]}) + g_{[0]}$$

Idea: 3rd-quantized observables

given by zero-form charges on \mathcal{G}

$$d_{\mathcal{G}} G \approx 0 \Rightarrow G \approx \text{Tr} \left(B_{[0]} * \tilde{B}_{[0]} \right)^{\otimes n}$$

\rightsquigarrow zero-form charges on $\Sigma \times M$.

$$\left(\text{using } \tau \rightsquigarrow B_{[0]} \overset{\tau_2}{\leftrightarrow} \tilde{B}_{[0]} \right)$$

Holographic defects in AdS₄

Konstein - Vasiliev mat₂ extension

$$\mathbb{X} \in \Omega_{\text{hor}}(\mathcal{X} \times \mathcal{M}) \otimes \text{mat}_2 \otimes \mathcal{G}_{\text{h}_{\text{gl}(1)}} \otimes \mathbb{C}[r]$$

Boundary conditions on S

$$\mathbb{X} \in \Omega(\mathcal{X} \times S') \otimes \mathcal{W} \otimes \mathcal{G}_{\text{h}_{\text{sp}(2,1)}} \otimes \mathbb{C}[r]$$

Fractional-spin algebra

$$\mathcal{W} = \left[\begin{array}{cc} \text{Con} \otimes \text{Con}^* & \text{Con} \otimes \text{Col}^* \\ \text{Col} \otimes \text{Con}^* & \text{Col} \otimes \text{Col}^* \end{array} \right]_{\text{even}}$$

Con = Supersingleton \oplus Anti-supersingleton
as a UR of conformal Sp(4)

= UIR of extended Weyl algebra \mathcal{W}

Col = Fock space \mathcal{F} as a UIR of colour $U(\infty)$ $\xrightarrow{\text{level truncate}}$ $U(N)$

$$\mathcal{W} \cong \left[\begin{array}{cc} \mathcal{W} & (\mathcal{F} \oplus \pi(\mathcal{F})) \otimes \mathcal{F}^* \\ \mathcal{F} \otimes (\mathcal{F} \oplus \pi(\mathcal{F}))^* & \mathcal{F} \otimes \mathcal{F}^* \end{array} \right]$$

Epimorphism
(Wigner-Ville)

$$\rho: \begin{aligned} \text{Con} \otimes \text{Col}^* &\rightarrow \text{Con} \otimes \text{Con}^* \\ \text{Col} \otimes \text{Con}^* &\rightarrow \text{Con} \otimes \text{Con}^* \end{aligned}$$

$$\begin{aligned} \mathbb{X} \times \mathbb{X}' &= \begin{bmatrix} p & \sigma \\ \bar{\sigma} & u \end{bmatrix} * \begin{bmatrix} p' & \sigma' \\ \bar{\sigma}' & u' \end{bmatrix} \\ &= \begin{bmatrix} p * p' + \rho(\sigma * \bar{\sigma}') & p * \sigma' + \sigma * u' \\ \bar{\sigma} * p' + u * \bar{\sigma}' & \bar{\sigma} * \sigma' + u * u' \end{bmatrix} \end{aligned}$$

Expand superconnection over $\mathcal{G}^h_{\text{gl}(1)}$

$$\mathbb{X} = \begin{bmatrix} A & B & \vdots & \psi & S \\ \tilde{B} & \tilde{A} & \vdots & \tilde{S} & \tilde{\psi} \\ \hline \tilde{\psi} & \tilde{S} & \vdots & V & M \\ S & \psi & \vdots & \tilde{M} & \tilde{V} \end{bmatrix}$$

$$\begin{aligned} A^\dagger &= -\tilde{A}, & B^\dagger &= B, & \tilde{B}^\dagger &= -\tilde{B} \\ \psi^\dagger &= -\tilde{\psi}, & \tilde{\psi}^\dagger &= -\psi \\ S^\dagger &= \tilde{S}, & \tilde{S}^\dagger &= -S \\ V^\dagger &= -\tilde{V}, & M^\dagger &= M, & \tilde{M}^\dagger &= -\tilde{M} \end{aligned}$$

Bulk Vasiliev branch

$$\begin{aligned} A &= \tilde{A} = W = -W^\dagger \\ \tilde{B} &= I = -I^\dagger \end{aligned} \quad \Rightarrow \quad \begin{cases} dW + W * W + B * I = 0 \\ dB + W * B - B * W = 0 \end{cases}$$

$N=1$

Defect fractional-spin branch (Nilsson-Vasiliev)

$$\begin{aligned} A &= \tilde{A} = W = -W^\dagger \\ V &= \tilde{V} = U = -U^\dagger \\ \tilde{S} &= I * S \in \text{Con} \otimes \text{Col}^* \\ \tilde{\tilde{S}} &= \tilde{S} * I \in \text{Col} \otimes \text{Con}^* \end{aligned} \quad \Rightarrow \quad \begin{cases} dW + W * W + \rho(S * \tilde{S} * I) = 0 \\ dU + U * U + \tilde{S} * I * S = 0 \\ dS + W * S - S * U = 0 \end{cases}$$

Bulk/defect correspondence $B \sim S * \tilde{S}$
(Flato-Fronsdal)

Summary of ideas:

1. Semi-classical higher-spin gravity \subset Non-linear quantum mechanics
2. Quantum higher-spin gravity \subset Non-commutative AKSZ sigma model
3. Holography subsumed into defect symplectic geometry $\xrightarrow{\text{AKSZ functor}}$ entangled operator algebras

Select remarks:

- Add T^*G to W-string \rightsquigarrow HS KM algebra
- Activate fiber in T^*G \leftarrow Conical singularities
- Left-right symmetric model on $T^*G \times S$ \rightsquigarrow quantum group
- Partial actions for $Z_{FSG} \rightleftharpoons Z_{HSG}$ \rightsquigarrow non-BPS holography
- Multi-singleton states \rightleftharpoons Vasiliev-Cherednik theory