

QUARKS ONLINE WORKSHOPS-2021  
Integrability, Holography, Higher-Spin Gravity and Strings  
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## Coadjoint representation of BMS4 on celestial Riemann surfaces

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## Context

- asymptotically flat gravity at null infinity  $J^+$
- symmetry group :  $\text{BMS}_4$
- why study coadjoint representation of  $\text{BMS}_4$  ?
- relevant for celestial holography , amplitudes , (semi-classical) QG

Classify coadjoint orbits (of  $\text{BMS}_4$ )

symplectic manifolds can be quantized  $\rightarrow$  relation to VIRREPS (Kirillov)

Write geometric actions Alekseev, Faddeev, Shatashvili J.Geom. Phys. 1988, Nucl. Phys. 1989

$$S = \int \text{Tr} (g^{-1})^\mu g \partial^\nu g^{-1} \partial_\nu g \) dx \rightarrow S = \int \left( \langle N_0^*, g^{-1} \frac{dg}{dt} \rangle - H \right) dt$$

no killing metric  
needed.

correct global  
symmetries

PI quantization  $\longrightarrow$  characters

for 3d gravity  $\Leftrightarrow$  to actions constructed from

GB, Gonzalez-Salguero CQG 2018

$CS \rightarrow W+W$

Eltzur et al. Nucl. Phys. 1989

Coussaert, Henneaux, Vandiehl CQG 1995

Effective actions for Goldstone bosons

# Contents

- Coadjoint representation for semi-direct product groups
- Construction for  $\text{BMS}_4$  on sphere & punctured plane
- Identification in asymptotically flat spacetimes at  $J^+$
- Perspectives

in collaboration with R. Ruzziconi, to appear JHEP 2021

(C.Troessaert, B. Oblak, P. Maor)

## Coadjoint representation & semi-direct product groups

adjoint  $f : [e_a, e_b] = f^c_{ab} e_c \quad (\text{ad } e_a)^b_c = f^b_{ac} \Leftrightarrow \text{ad } e_a(e_b) = f^c_{ab} e_c$

coadjoint  $f^* : \langle e_*^b, e_a \rangle = \delta_a^b \quad (\text{ad}^* e_a) = -(\text{ad } e_a)^T \Leftrightarrow \text{ad}^* e_a(e_*^b) = -f^b_{ac} e_*^c$

group  $\text{Ad}_g e_a = g e_a g^{-1}, \quad \text{Ad}_g^* = g e_*^b g^{-1}$

semi-direct product  $G \times_A \underset{\tau}{\lrcorner} A : (f, \alpha) \cdot (g, \beta) = (f \cdot g, \alpha + \tau_f(\beta))$  \tau: representation  
 $\text{ISO}(3), \text{ISO}(3,1),$  A: abelian ideal  
 $\text{BMS}_3, \text{BMS}_4 \dots$   $g \oplus_{\Sigma} A : [(X, \alpha), (Y, \beta)] = ([X, Y], \sum_x \beta - \sum_y \alpha)$

$$\text{Ad}_{(f, \alpha)} (X, \beta) = (\text{Ad}_f X, \tau_f \beta - \sum_{\alpha} \beta - \sum_{\alpha} \alpha)$$

$$\text{ad}_{(X, \alpha)} (Y, \beta) = ([X, Y], \sum_x \beta - \sum_y \alpha)$$

dual space  $\mathfrak{g}^* \oplus A^*$   $\langle (j, p), (x, \alpha) \rangle = \langle j, x \rangle + \langle p, \alpha \rangle$

terminology  $j$ : angular momentum  $p$ : linear momentum  
 $x$ : inf. rotation  $\alpha$ : inf. translation

BMS: odd "super"

ingredients  $x : A \oplus A^* \rightarrow \mathfrak{g}^* : \langle \alpha \times p, x \rangle = \langle p, \sum_x \alpha \rangle$

change in angular momentum due to  $\alpha$  translation

$$\tau^* : G \times A^* \rightarrow A^* : \langle \tau_f^* p, \alpha \rangle = \langle p, \tau_{f^{-1}}^* \alpha \rangle$$

coadjoint representation

$$\text{Ad}_{(f, \alpha)}^*(j, p) = (\text{Ad}_f^* j + \alpha \times \tau_f^* p, \tau_f^* p)$$

$$\text{ad}_{(x, \alpha)}^*(j, p) = (\text{ad}_x^* j + \alpha \times p, \sum_x^* p)$$

# Poincaré & $BMS_4$ algebra at $\mathbb{J}^+$

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Poincaré generators  $\mathbb{R}^{3,1}$        $L^{ab} = -\left(x_a^{\alpha} \frac{\partial}{\partial x_b} - x_b^{\alpha} \frac{\partial}{\partial x_a}\right)$ ,     $P^a = \frac{\partial}{\partial x^a}$        $\eta_{ab} = \text{diag}(1, -1, -1, -1)$

structure constants

$$\begin{cases} [L^{ab}, L^{cd}] = -(\eta^{bc} L^{ad} - \eta^{ac} L^{bd} - \eta^{bd} L^{ac} + \eta^{ad} L^{bc}) \\ [P^a, L^{bc}] = -(\eta^{ab} P^c - \eta^{ac} P^b) \end{cases}$$

Boost & rotation generators

$$\begin{cases} L_z = L^{12}, & L^{\pm} = \pm i(L^{23} + L^{13}) \\ H = P^0, & P_z = -\frac{1}{2} P^3, \quad P^{\pm} = -\frac{1}{2} (i P^2 \mp P^1) \end{cases}; \quad K_z = L^{30}, \quad K^{\pm} = \mp i L^{20} - L^{10}$$

structure constants

$$[L^+, L^-] = 2iL_z, \quad [L_z, L^{\pm}] = \pm iL^{\pm}, \quad \dots$$

spherical & retarded time  $r = \sqrt{\sum_i (x^i)^2}$ ,  $u = x^0 - r$ ,  $r \cos \theta = x^3$ ,  $r \sin \theta e^{i\phi} = x^1 + i x^2$

$$L_z = J_\phi, L^\pm = e^{\pm i\phi} [J_0 \pm i \cot \theta J_\phi], K_z = -(1 + \frac{u}{r}) \cos \theta r J_\theta + \cos \theta (u) u + (1 + \frac{u}{r}) \sin \theta J_\phi$$

$$K^\pm = e^{\pm i\phi} [(1 + \frac{u}{r}) \sin \theta r J_\theta - \sin \theta (u) u + (1 + \frac{u}{r}) \cos \theta J_\phi \pm (1 + \frac{u}{r}) \frac{i}{r} \sin \theta J_\phi]$$

$$H = J_u, -2P_z = \cos \theta (-\cancel{r} + J_u) + \frac{1}{r} \sin \theta J_\theta, \pm 2P^\pm = e^{\pm i\phi} [\sin \theta (\cancel{r} - J_u) + \frac{1}{r} \cos \theta J_\theta \pm \frac{1}{r} \sin \theta J_\phi]$$

Simplification 1:  $J^+$   $r = cte \Rightarrow \varpi$

Simplification 2: cut  $u=0$  of  $J^+$

$$K_z = \sin \theta J_\theta, K^\pm = e^{\pm i\phi} [\cos \theta J_\theta \pm \frac{i}{\sin \theta} J_\phi]$$

$$H = 1 \sim {}_0Y_{00}, P_z = -\frac{1}{2} \cos \theta \sim {}_0Y_{10}, J^\pm = \mp \frac{1}{2} e^{\pm i\phi} \sin \theta \sim {}_0Y_{1\pm 1}$$

4 lowest harmonics

Poincaré algebra

$$[L_z, f] = L_z(f), [L^\pm, f] = L^\pm(f)$$

$$f = H, P_z, P^\pm$$

$$[K_z, f] = K_z(f) - \cos \theta f, [K^\pm, f] = K^\pm(f) + e^{\pm i\phi} \sin \theta f$$

BMS<sub>4</sub> algebra:

$$f \in C^\infty(S^2)$$

Sachs Phys Rev 1962

Simplification 3: stereographic coordinates on the sphere

$$\xi = \cot \frac{\theta}{2} e^{-i\phi}$$

$$ds^2 = -2(P_S \bar{P}_S) d\xi d\bar{\xi}$$

$$P_S = \frac{1}{2R^2} (\lambda + \xi \bar{\xi})$$

Lorentz algebra

$$l_m = \xi^{1-m} j, \quad \bar{l}_m = \bar{\xi}^{1-m} \bar{j}$$

$$sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$$

$$L_z = -i(l_0 - \bar{l}_0), \quad K_z = -(l_0 + \bar{l}_0), \quad L^+ = l_1 + \bar{l}_{-1}, \quad L^- = \bar{l}_1 + l_{-1}, \quad K^+ = -(\bar{l}_1 - l_1), \quad K^- = -(l_{-1} - \bar{l}_1)$$

action of Lorentz on (super)-translations

$$H = 1, \quad P_z = \frac{1 - \xi \bar{\xi}}{2(\lambda + \xi \bar{\xi})}, \quad P^+ = -\frac{\bar{\xi}}{\lambda + \xi \bar{\xi}}, \quad P^- = \frac{\xi}{\lambda + \xi \bar{\xi}}$$

$$[K_z, f] = K_z(f) + \frac{1 - \xi \bar{\xi}}{\lambda + \xi \bar{\xi}} f \quad [K^+, f] = K^+(f) + \frac{2 \bar{\xi}}{\lambda + \xi \bar{\xi}} f, \quad [K^-, f] = K^-(f) + \frac{2 \xi}{\lambda + \xi \bar{\xi}} f$$

## Coadjoint representation of $\text{BMS}_4$ : general structure

2d conformally flat S zim: unified description for sphere & punctured plane

$$ds^2 = -2(\bar{P}\bar{\bar{P}})^{-1} d\xi d\bar{\xi} \quad \left\{ \begin{array}{l} \xi' = \xi'(\xi) \quad \bar{\xi}' = \bar{\xi}'(\bar{\xi}) \\ P'(x) = P(x)e^{-E(x)}, \quad \bar{P}'(x) = \bar{P}(x)e^{-\bar{E}(x)} \end{array} \right. \quad \text{conformal coordinate transf.}$$

$$x = (\xi, \bar{\xi})$$

$$\text{complex Weyl rescaling}$$

zweiheits

$$ds^2 = e^\alpha \partial x^\mu \eta_{\mu\nu} e^\beta \partial x^\nu \quad \eta_{\mu\nu} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\ell_1^\mu \frac{\partial}{\partial x^\mu} = P \frac{\partial}{\partial \xi}$$

$$\ell_2^\mu \frac{\partial}{\partial x^\mu} = \bar{P} \frac{\partial}{\partial \bar{\xi}}$$

conformal fields

$$\phi_{\alpha, \bar{\alpha}}^{(1)}(x') = \left( \frac{\partial \xi}{\partial \xi'}, \frac{\partial \bar{\xi}}{\partial \bar{\xi}'} \right) \bar{w} \phi_{\alpha, \bar{\alpha}}(x)$$

$$P'(x') = P(x) e^{-E(x')} \frac{\partial}{\partial \xi'}$$

combined transf.

weighted scalars

$$\eta^{(s,w)}(x') = e^{w E_R(x')} e^{-is E_I(x')} \eta^{s,w}(x)$$

Held, Pashas, Newman JMP 1970

interpolation

$$\eta^{s,w} = P^w \bar{P}^{\bar{w}} \phi_{\alpha, \bar{\alpha}}$$

Lorentz group & sphere

map

$$S = h - \bar{h}, \quad w = -(h + \bar{h})$$

$$h = \frac{s-w}{2}, \quad \bar{h} = -\frac{s+w}{2}$$

D'Hoker, Phong Rev. Mod. Phys. 1988

covariant derivative

$$\nabla : \quad \Gamma_{\xi\xi}^{\xi} = - J \ln(P\bar{P}) \quad \Gamma_{\bar{\xi}\bar{\xi}}^{\bar{\xi}} = - \bar{J} \ln(\bar{P}\bar{\bar{P}})$$

$$\Gamma_{\xi'\xi'}^{\xi'}(x') = \Gamma_{\xi\xi}^{\xi}(x) \frac{\partial \xi}{\partial \xi'} + \frac{\partial \xi'}{\partial \xi} \frac{\partial^2 \xi}{\partial \xi' \partial \xi'} + 2 J' E_R(x')$$

introduce Weyl connection  $\mathcal{D}$ :  $W'(x') = \frac{\partial \xi}{\partial \xi'} W + 2 J' E_R(x'), \bar{W}'(x') = \frac{\partial \bar{\xi}}{\partial \bar{\xi}} \bar{W}(x) + 2 \bar{J}' \bar{E}_R(x')$

$$\underbrace{\mathcal{D}_{\xi} \phi_{h,\bar{w}}}_{(h+1, \bar{h})} = [\nabla + h W] \phi_{h,\bar{w}}, \quad \underbrace{\bar{\mathcal{D}}_{\bar{\xi}} \phi_{h,\bar{w}}}_{(\bar{h}, \bar{h}+1)} = [\bar{\nabla} + \bar{h} W] \phi_{h,\bar{w}}$$

$$J, \nabla, D = J_\xi, D_\xi, D_{\bar{\xi}}$$

$$\bar{J}, \bar{\nabla}, \bar{D} = \bar{J}_{\bar{\xi}}, \bar{D}_{\bar{\xi}}, \bar{D}_{\bar{\xi}}$$

weighted scalars

$$\begin{aligned} \mathcal{D}_{\xi} \eta^{s,w} &= P^{h+1} \bar{P}^{\bar{h}} \nabla \phi_{h,\bar{w}}, & \bar{\mathcal{D}}_{\bar{\xi}} \eta^{s,w} &= \bar{P}^h \bar{P}^{h+1} \bar{\nabla} \phi_{h,\bar{w}} \\ &= P \bar{P}^{-s} J(\bar{P}^s \eta^{s,w}) & &= \bar{P} P^s \bar{J}(P^{-s} \eta^{s,w}) \end{aligned}$$

Weyl covariant

$$\begin{aligned} \underbrace{\mathcal{D}_{\xi} \eta^{s,w}}_{[s+1, w-1]} &= P^{h+1} \bar{P}^{\bar{h}} \mathcal{D}_{\xi} \phi_{h,\bar{w}}, \\ &= \left( J + \left( \frac{s-w}{2} \right) PW \right) \mathcal{D}_{\xi} \phi_{h,\bar{w}} \end{aligned}$$

$$\begin{aligned} \underbrace{\bar{\mathcal{D}}_{\bar{\xi}} \eta^{s,w}}_{[s-1, w-1]} &= P^h \bar{P}^{h+1} \bar{\mathcal{D}}_{\bar{\xi}} \phi_{h,\bar{w}} \\ &= \left[ \bar{J} - \left( \frac{w-s}{2} \right) \bar{P} \bar{W} \right] \phi_{h,\bar{w}} \end{aligned}$$

$$[\mathcal{D}, \bar{\mathcal{D}}] \eta^{s,w} = - \frac{s}{2} R_S \eta^{s,w} - P \bar{P} \left( \frac{s-w}{2} JW + \frac{s+w}{2} J \bar{W} \right) \eta^{s,w}$$

$R_S$ : scalar curvature

## Ingredients

(super-)translation  $T : [0, 1]$   $\tilde{T} : (-\frac{1}{2}, -\frac{1}{2})$  real

(super-)rotation  $y : [-1, 1]$   $\tilde{y} : (-1, 0)$   $\bar{D}y = 0 \Leftrightarrow \bar{D}\tilde{y} = 0$   
 $\bar{y} : [1, 1]$   $\tilde{\bar{y}} : (0, -1)$   $D\bar{y} = 0 \Leftrightarrow D\tilde{\bar{y}} = 0$

(super-)momentum  $P : [0, -8]$   $\tilde{P} : (\frac{3}{2}, \frac{3}{2})$  real

(super-)sugardr momentum  $J : [-1, -3]$   $\tilde{J} : (1, 2)$   $J \sim J + \bar{D}\bar{L}$ ,  $\tilde{J} \sim \tilde{J} + \bar{D}\tilde{\bar{L}}$   
 $\bar{J} : [1, -3]$   $\tilde{\bar{J}} : (2, 1)$   $\bar{J} \sim \bar{J} + \bar{D}\bar{L}$ ,  $\tilde{\bar{J}} \sim \tilde{\bar{J}} + \bar{D}\tilde{\bar{L}}$   
 $(-2, -2)$   $(0, 2)$   
 $[2, -2]$   $(2, 0)$

In all relations, weights/dimensions are such that Weyl connection drops out!

$D \rightarrow S$      $D \rightarrow J$     simplest description in terms of conformal fields

bms<sub>4</sub> algebra

$$[(y_1, \bar{y}_1, J_1), (y_2, \bar{y}_2, J_2)] = (\hat{y}, \frac{1}{\bar{y}}, \hat{J})$$

$$\hat{y} = y_1 J y_2 - y_2 J y_1$$

$$\hat{J} = y_1 J y_2 - \frac{1}{2} J y_1 J y_2 - (l \leftrightarrow r) + c.c.$$

subalgebra of  
(Lorentz, Witt $\oplus$ Witt)

$$(y, \bar{y}, 0)$$

$$(\tilde{y}, \bar{\tilde{y}}, 0)$$

representation of  $g$  on  $\eta^{s,w}$  on  $\phi_{h,\bar{h}}$

$$y \cdot \eta^{s,w} = y J \eta^{s,w} + \frac{s-w}{2} J y \eta^{s,w}$$

$$\tilde{y} \cdot \phi_{h,\bar{h}} = \tilde{y} J \phi_{h,\bar{h}} + h J \tilde{y} \phi_{h,\bar{h}}$$

$$\bar{y} \cdot \eta^{s,w} = \bar{y} \bar{J} \eta^{s,w} - \frac{s+w}{2} \bar{J} \bar{y} \eta^{s,w}$$

$$\bar{\tilde{y}} \cdot \phi_{h,\bar{h}} = \bar{\tilde{y}} \bar{J} \phi_{h,\bar{h}} + \bar{h} \bar{J} \bar{\tilde{y}} \phi_{h,\bar{h}}$$

$$\Sigma_{x^2} = (y, \bar{y}) \cdot J^{[0,1]}$$

$$\Sigma_{x^4} = (\tilde{y}, \bar{\tilde{y}}) \cdot \bar{J}^{(-1_k, -1_k)}$$

action of inf rotation on foundations

$$dms_4^* \quad \text{dual space} \quad ([\bar{J}], [\bar{\bar{J}}], \bar{P}) \quad ([\tilde{J}], [\tilde{\bar{J}}], \tilde{\bar{P}}) \quad (0,0) ; [0, -2]$$

pairing  $\langle ([\bar{J}], [\bar{\bar{J}}], \bar{P}); (\bar{y}, \bar{\bar{y}}, \bar{T}) \rangle = \int_S d\mu [\bar{J}\bar{y} + \bar{\bar{J}}\bar{\bar{y}} + \bar{P}\bar{T}]$ ,  $d\mu(\xi, \bar{\xi}) = \frac{iC}{PP} d\xi_1 d\bar{\xi}$

$$\langle ([\tilde{J}], [\tilde{\bar{J}}], \tilde{\bar{P}}), (\tilde{y}, \tilde{\bar{y}}, \tilde{T}) \rangle = \int_S d\tilde{\mu} [\tilde{J}\tilde{y} + \tilde{\bar{J}}\tilde{\bar{y}} + \tilde{\bar{P}}\tilde{T}] \quad d\tilde{\mu} = iC d\xi_1 d\bar{\xi}$$

assumption : pairing annihilates total  $\bar{J}, \bar{\bar{J}} (J, \bar{J})$  derivatives  
 non-degenerate  $\rightarrow$  integrations by parts

$$ad^*_{(y, \bar{y}, T)} J = \bar{y} \bar{J} J + 2 \bar{J} \bar{y} J + \underbrace{J(yJ)}_{= ad^*_{\bar{y}} J \sim 0} + \underbrace{\frac{1}{2} T \bar{J} P + \frac{3}{2} \bar{J} J P}_{\propto P},$$

$$ad^*_{(y, \bar{y}, T)} P = \underbrace{y J P + \frac{3}{2} J y P}_{E_x^* P} + \text{c.c.}$$

work out formulas for  
 the group ✓

## Realization on the sphere

stereographic coord.

$$\xi = \cot \frac{\theta}{2} e^{-i\phi}$$

$$d\xi^2 = -2(P_S \bar{P}_S) d\xi d\bar{\xi}$$

$$P_S = \frac{1}{R^2} (\lambda + \xi \bar{\xi})$$

globally well-defined conf.

coord. for f

$$\xi' = \frac{a\xi + b}{c\xi + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C} \quad \frac{\partial \xi}{\partial \xi'} = (c\xi + d)^2$$

compensating Weyl fuf.

$$e^{F_R(x')} = \frac{1 + \bar{\xi}\xi}{|a\xi + b|^2 + |c\xi + d|^2}$$

$$e^{F(x')} = \frac{\bar{c}\bar{\xi} + \bar{d}}{c\xi + d}$$

w: boost weight

Pairing  $\langle K^{S_i-w-2}, M^{S_i w} \rangle = \frac{1}{4\pi R^2} \int_{S^2} i \overline{\frac{d\xi \wedge d\bar{\xi}}{P_S \bar{P}_S}} \overline{K^{S_i-w-2}} \eta^{S_i w}$

$$C = (4\pi R^2)^{-1}$$

assumptions ✓

$$\frac{1}{4\pi R^2} \int_{S^2} i \overline{\frac{d\xi \wedge d\bar{\xi}}{P_S \bar{P}_S}} = \frac{1}{4\pi} \int_0^\pi \int_0^\pi \int_0^{2\pi} \partial\theta \sin\theta \partial\phi = 1$$

adjoint repres. group

$$y'(x') = e^{\frac{1}{2}F_R(x')} e^{i\frac{1}{2}E_I(x')} y(x)$$

$$\beta'(x') = e^{\frac{1}{2}F_R(x')} \left( \beta - (\gamma f_2 - \frac{1}{2} f_2 \bar{f} \gamma + \text{c.c.}) \right) (x)$$

cadjoint repres. group

$$J'(x') = e^{-\frac{3}{2}F_R(x')} e^{i\frac{1}{2}E_I(x')} (J^+ - \frac{1}{2} J \bar{f} P^+ - \frac{3}{2} \bar{f} J P^+) (x)$$

$$P'(x') = e^{-\frac{3}{2}F_R(x')} P(x)$$

in terms of conf. fields

$$\tilde{y}'(s') = (c s + d)^{-2} \tilde{y}(s)$$

$$\tilde{\beta}'(x') = (c s + d)^{-4} (\bar{c} \bar{s} + \bar{d})^{-4} \left( \tilde{\beta} - \tilde{y} f_2 - \frac{1}{2} \bar{d} \bar{f} \tilde{y} + \text{c.c.} \right) (x)$$

$$\tilde{J}'(x') = (c s + d)^3 (\bar{c} \bar{s} + \bar{d})^4 \left( \tilde{J}(x) + \left( \frac{1}{2} \tilde{T} \bar{J} \tilde{P}^+ + \frac{3}{2} \bar{T} \tilde{J} \tilde{P}^+ \right) (x) \right)$$

$$\tilde{P}'(x') = (c s + d)^3 (\bar{c} \bar{s} + \bar{d})^3 \tilde{P}(x)$$

Expansions : spin weighted spherical harmonics  $s^z_{jm}$  unnormalized  
 $s^y_{jm}$  normalized

conformal Killing eq. on  $S^2$

$$\bar{\mathcal{L}}\mathcal{Y}^{[-1,1]} = 0 = \mathcal{L}\bar{\mathcal{Y}}^{[1,-1]}$$

$$y_m = -R\sqrt{2} {}_{-1}\mathcal{Z}_{1,m} \quad m = -1, 0, 1$$

$$Y = \sum_{m=-1}^1 y_m Y_m$$

$$\mathcal{T}_{jm} = {}_0\mathcal{Z}_{jm}$$

$$\mathcal{T} = \sum_{j, l, m \leq j} t_{jm} \mathcal{T}_{jm}, \quad \bar{t}_{jm} = (-1)^m t_{j-m}$$

dual basis

$$y_*^m = \frac{-i}{R\sqrt{2}(l+m)! (l-m)!} {}_{-1}\mathcal{Z}_{1,m}$$

$$\bar{Y} = \sum_{m=-1}^1 j_{jm} Y_*^m$$

$$\mathcal{T}_*^{jm} = \frac{(2j+1)! (2j)!}{j! j! (j+m)! (j-m)!} {}_0\mathcal{Z}_{jm}$$

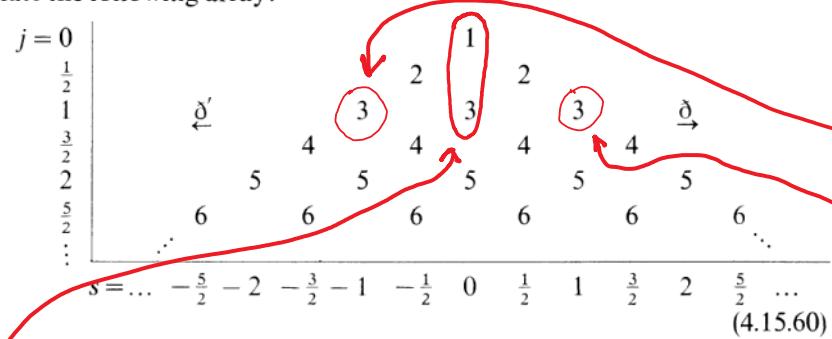
$$\mathcal{P} = \sum_{j, l, m \leq j} p_{jm} \mathcal{T}_*^{lm}, \quad \bar{p}_{jm} = (-1)^m p_{j-m}$$

NB: conformal fields :  $\tilde{Y}_m = Y_m P_S = S^{1-m} \Rightarrow [\tilde{Y}_m, \tilde{Y}_n] = (m-n) \tilde{Y}_{m+n}$

→ all other structure constants can be worked out explicitly (ugly)

## Remark (i) Penrose & Rindler Vol I, section 4.15

In the study of spin-weighted spherical harmonics it is useful to contemplate the following array:



The numbers in this triangular array (which extends indefinitely downwards) represent the complex *dimensions* of the various spaces of spin-weighted spherical harmonics, as discussed in (4.15.43) *et seq.* Each of these spaces is characterized by its values of  $s$  and  $j$ , as shown. The dimension zero is assigned wherever a blank space appears in the array. The operator  $\delta$  carries us a step of one  $s$ -unit to the right and  $\delta'$  one  $s$ -unit to the left. (From our earlier discussion, the  $j$ -value is not affected by  $\delta$  or  $\delta'$ .) Whenever such a step carries us off the array, the result of the operator  $\delta$  or  $\delta'$  is zero. Note that the dimension remains constant whenever it does not drop to, or increase from, zero.

$$w \geq |s| \quad \mathcal{J}^{w+s+1} M^{s,w}$$

$$[w+1, s-1]$$

definite boost weight

$$\bar{\mathcal{J}}\mathcal{Y} = 0 \Leftrightarrow \mathcal{J}^3 \mathcal{Y} = 0$$

$$\mathcal{J}\bar{\mathcal{Y}} = 0 \Leftrightarrow \bar{\mathcal{J}}^3 \bar{\mathcal{Y}} = 0$$

same solutions

dual situation  $w \leq -|s|-2$

$$\mathcal{J}^{s-w-1} M^{w+1, s-1}$$

$$[s, w]$$

$$\bar{\mathcal{J}}^{-s-w-1} \bar{M}^{-w-1, -s-1}$$

$[s, w]$  definite boost weight

$$\bar{\mathcal{Y}} \sim \bar{\mathcal{J}} + \bar{\mathcal{J}} \bar{M} \quad \Leftrightarrow \quad \bar{\mathcal{Y}} \sim \bar{\mathcal{J}} + \mathcal{J}^3 M$$

$$[-2, 2]$$

$$[-2, 0]$$

same equivalence classes

## Remark (ii) reduction to Poincaré

$$\mathcal{F}^2 \mathcal{T} = 0 = \bar{\mathcal{F}}^2 \mathcal{T} \quad \mathcal{P} \sim \mathcal{P} + \mathcal{F}^2 M + \bar{\mathcal{F}}^2 \bar{M}$$

$$[-1, 1] \quad [2, -1]$$

Remark (iii)

$$\tilde{f}_m \cdot \phi_{\mu, \bar{\nu}} = \xi^{-m} (\xi) \phi_{\mu, \bar{\nu}} + \mu(\ell-m) \phi_{\mu, \bar{\nu}}$$

$$\bar{\tilde{f}}_m \cdot \phi_{\mu, \bar{\nu}} = \bar{\xi}^{-m} (\bar{\xi}) \phi_{\mu, \bar{\nu}} + \bar{\nu}(\ell-m) \phi_{\mu, \bar{\nu}}$$

Goldberg et al. JMP 1967

$$s^y_{j,m} \quad j \leq L \quad \longleftrightarrow \quad s^z_{m_1, m_2}^L = (1 + \xi \bar{\xi})^{-L} \xi^{L-s-m_1} \bar{\xi}^{L+s-m_2}$$

invertible

overcomplete set  
of functions,

look like expansions on the  
punctured plane

when transforming to associated conformal fields  
structure constants look like those on the  
punctured plane, up to corrections.

## Realization on punctured plane

- Weyl trsf  $e^{-\mathcal{F}(\xi, \bar{\xi})} = \frac{\Omega}{\lambda + \xi \bar{\xi}}$      $\xi = R^z z$      $d\Omega^2 = -2 dz d\bar{z}$

- 2-punctures: remove points at origin & infinity  $\mathbb{C}_0$

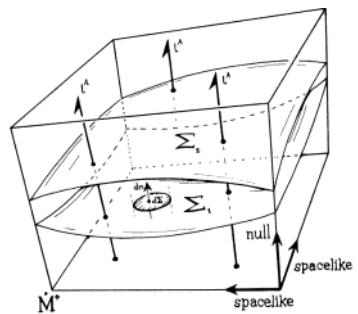
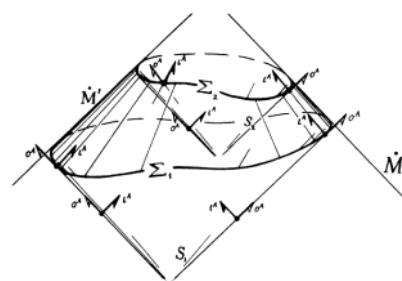
- on the level of the algebras, look at the algebra of all infinitesimal local conformal trsf.

Not the Lie algebra of globally well-defined trsf.

$P=1 \Rightarrow$  weighted scalars = conformal fields

$$e^{\mathcal{E}(x)} = \frac{\partial z}{\partial z} \quad e^{\mathcal{F}_R(x)} = \left( \frac{\partial z}{\partial t} \frac{\partial \bar{z}}{\partial \bar{t}} \right)^{1/2}, \quad e^{\mathcal{F}_I(x)} = \left( \frac{\partial z}{\partial \bar{t}} / \frac{\partial \bar{z}}{\partial t} \right)^{1/2}$$

For asymptotically flat spaces,  $\hat{M}$  is in fact a null hypersurface [7]. The structure of  $\hat{M}$  is essentially the same as for Minkowski space (Figure 4). We shall omit the three points  $I^-$ ,  $I^0$ ,  $I^+$  here. Then  $\hat{M}$  consists of two portions, each of which is topologically a "cylinder"  $S^2 \times E^1$ . We are concerned, here, only with the future portion  $\hat{M}^+$ , and by judicious choice of conformal factor  $\Omega$ , we can ensure that the geometry of  $\hat{M}^+$  is as simple as possible. In fact, by taking one generator of  $\hat{M}^+$  "back to infinity" we can open out the cylinder into a space with Euclidean three-space topology. Furthermore, it turns out that we can also make this three-space metrically flat (Figure 6). This will simplify matters considerably.



Penrose 1967 AMS

gravity: sphere  $\rightarrow$  plane

CFT : plane  $\rightarrow$  sphere

Conformal glc?

## Expansions

$$\phi_{\bar{h}\bar{l}}(z, \bar{z}) = \sum_{k,l} \alpha_{k,l} \overset{\sim}{h\bar{h}} z^k \bar{z}^l, \quad \overset{\sim}{h\bar{h}} \overset{\sim}{t}_{k,l} = z^{-h-k} \bar{z}^{-\bar{h}-l}$$

$h, \bar{h} \in \mathbb{N} \Rightarrow k, l \in \mathbb{Z}$   
 $h, \bar{h} \in \frac{\mathbb{N}}{2} \Rightarrow k, l \in \frac{1}{2} + \mathbb{Z}$   
 (NS)

Pairing  $\langle \psi_{-\bar{h}+1, -h+1}, \phi_{h, \bar{h}} \rangle = \text{Res}_z \text{Res}_{\bar{z}} [\overline{\psi_{-\bar{h}+1, -h+1}} \phi_{h, \bar{h}}]$

Assumptions ✓

$$\text{Res}_z (\mathcal{J}\phi) = 0 = \text{Res}_{\bar{z}} (\bar{\mathcal{J}}\phi)$$

adjoint repr. group  $\tilde{y}'(z') = \left(\frac{\partial z}{\partial z'}\right)^{-1} \tilde{y}(z)$

$$\tilde{\beta}'(x') = \left(\frac{\partial z}{\partial z'}\right)^{-1/2} \left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right)^{-1/2} \left(\tilde{\beta} - (\tilde{y})_x^2 - \frac{1}{2} \tilde{d} \tilde{J} \tilde{y} + \text{c.c.}\right)(x)$$

coadjoint repr group  $\tilde{J}'(x') = \left(\frac{\partial z}{\partial z'}\right)^1 \left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right)^2 \left(J + \frac{1}{2} \tilde{J} \tilde{J} \tilde{F} + \frac{1}{2} \tilde{J} \tilde{J} \tilde{P}\right)(x)$

$$\tilde{F}'(x) = \left(\frac{\partial z}{\partial z'}\right)^{3/2} \left(\frac{\partial \bar{z}}{\partial \bar{z}'}\right)^{3/2} \tilde{F}(x)$$

to be used for conformal mapping.

Expansions

$$\langle \tilde{\mathcal{Z}}_{k+l}, \tilde{\mathcal{Z}}_{k,l} \rangle = \delta_{l+k}^0 \delta_{k+l}^0$$

$$\tilde{f}_m = z^{1-m}, \quad \tilde{T}_{k,l} = z^{12-k} \bar{z}^{12-l} \quad m, \frac{l}{2} + k, \frac{k}{2} + l \in \mathbb{Z}$$

$$\tilde{f}_*^m = z^{-1} \bar{z}^{-2+m} \quad \tilde{T}_*^{k,l} = z^{-\frac{3}{2}+l} \bar{z}^{-\frac{3}{2}+k}$$

$$\tilde{f}_m \cdot \tilde{\mathcal{Z}}_{k,l} = -(\mu_m + \alpha) \tilde{\mathcal{Z}}_{k+\mu_m, l}, \quad \tilde{f}_m \cdot \tilde{\mathcal{Z}}_{k,l} = -(\bar{\mu}_m + l) \tilde{\mathcal{Z}}_{k, l+\mu_m}$$

Structure constants  $[\tilde{f}_m, \tilde{f}_n] = (\mu_m - \mu_n) \tilde{f}_{m+n} \quad [\tilde{f}_m, \tilde{T}_{k,l}] = \left(\frac{1}{2}\mu_m - \alpha\right) \tilde{T}_{k+\mu_m, l}$

$$[\tilde{f}_m, \tilde{T}_{k,l}] = \left(\frac{1}{2}\mu_l - l\right) \tilde{T}_{k, l+\mu_m}$$

$$[\tilde{f}_m, \tilde{f}_n] = 0 = [\tilde{T}_{k,l}, \tilde{T}_{n,s}]$$

coadjoint repr. algebra

$$\text{ad}^*_{\tilde{Y}_m} \tilde{Y}_n^{\mu} = (-2m+n) \tilde{Y}_n^{\mu-m}, \quad \text{ad}^*_{\tilde{Y}_m} \tilde{T}_{*}^{k,l} = \left(-\frac{3}{2}m+k\right) \tilde{T}_{*}^{k-m,l}$$

$$\text{ad}^*_{\tilde{T}_{k,l}} \tilde{T}_{*}^{r,s} = \left(\frac{r-3k}{2}\right) \delta_r^s Y_n^{r-k} + \left(\frac{s-3l}{2}\right) \delta_k^r \tilde{Y}_n^{s-l}$$

Realization on cylinder

$$z = e^{-i \frac{2\pi}{L_1} w}, \quad w = w_1 + i w_2, \quad w_1 \sim w_1 + L_1, \quad \phi_{u,\bar{u}}^c(w, \bar{w}) = \left(i \frac{2\pi}{L_1} z\right)^{\bar{u}} \left(i \frac{2\pi}{L_1} \bar{z}\right)^{\bar{u}} \phi_{u,\bar{u}}(z, \bar{z})$$

use formulas for the group to map generators

$$Y_m^c = i \left(\frac{2\pi}{L_1}\right)^{-1} e^{i \frac{2\pi}{L_1} mw}$$

same structure constants, obtained from  $\text{ad}^*$  still provide a representation

but pairing issues ...

TORUS:

$$\begin{pmatrix} & & \\ w_2^T \sim w_2^T + L_2 & \xrightarrow{e^{i \frac{2\pi}{L_1}(w_1 + i w_2)}} & e^{i \frac{2\pi}{L_1} w_1^T} e^{i \frac{2\pi}{L_2} w_2^T} \\ & & \\ & & w_2 = +i \frac{L_2}{L_1} w_2^T \end{pmatrix}$$

## Identification in non-radiative asymptotically flat spacetimes at $\mathcal{J}^+$

Back to  $S^2 \times GR$ : BMS metric  $\Leftrightarrow$  NP first order

Solution space, free data at  $\mathcal{J}^+$ :  $\psi_2^\circ + \bar{\psi}_2^\circ, \psi_1^\circ, \tau^\circ$  undetermined  $u$ -dependence  
 $\dot{\tau}^\circ$  news

evolution equations:  $\Im u \psi_3^\circ = \mathcal{J} \psi_3^\circ + \tau^\circ \psi_4^\circ, \quad \Im u \psi_1^\circ = \mathcal{J} \psi_2^\circ + 2\tau^\circ \psi_3^\circ$

constraints  $\psi_2^\circ - \bar{\psi}_2^\circ = \bar{\mathcal{J}}^2 \tau^\circ - \mathcal{J}^2 \bar{\tau}^\circ + \dot{\tau}^\circ \bar{\tau}^\circ - \tau^\circ \dot{\bar{\tau}}^\circ$   
 $\psi_3^\circ = -\mathcal{J} \dot{\bar{\tau}}^\circ, \quad \psi_4^\circ = -\ddot{\bar{\tau}}^\circ$

additional data to construct solutions  
 $\psi_0 = \sum_{n \geq 0} \psi_n^u (\xi, \bar{\xi}, u_0) r^{-5-n}$

Transformation of (relevant) free data at  $\mathcal{T}^*$

$$s = (y, \psi, \bar{\psi}), \quad f = \mathcal{T} + \frac{1}{2} u (\bar{\psi} \gamma + \bar{\bar{\psi}} \bar{\gamma})$$

$$\delta_s \mathcal{T}^* = [f J_u + \psi \bar{\psi} + \bar{\psi} \bar{\bar{\psi}} + \frac{3}{2} \bar{\psi} \psi - \frac{1}{2} \bar{\psi} \bar{\psi}] \mathcal{T}^* - f^2 f$$

$$\delta_s \psi_2^* = [u \bar{u} \bar{u} + \frac{3}{2} \bar{\psi} \psi + \frac{3}{2} \bar{\psi} \bar{\psi}] \psi_2^* + 2 \bar{\psi} f \psi_3^*$$

(constraints to be imposed)

$$\delta_s \psi_1^* = [u \bar{u} \bar{u} + 2 \bar{\psi} \psi + \bar{\psi} \bar{\psi}] \psi_1^* + 3 \bar{\psi} f \psi_2^*$$

broken current

algebra

$$J_s = \frac{i}{\hbar^2} \left[ (P_s \bar{P}_s)^{-1} \bar{J}_s^u \partial \xi_1 d\bar{\xi} + P_s^{-1} \bar{J}_s^{\bar{\xi}} \partial u_1 d\xi - \bar{P}_s \bar{J}_s^{\xi} \partial u_1 d\bar{\xi} \right]$$

$$\delta_{s_1} J_{s_2} + \boxed{\Theta_{s_2}(\delta_{s_1} X)} \approx - J_{\{s_1, s_2\}} + d L_{s_1, s_2}$$

non-conservation

$$d J_s + \Theta_s(\delta_{(0,0,1)} X) \approx 0$$

$$s_n: (y, \psi, \bar{\psi}) = (0, 0, 0)$$

$\Theta_s(\delta X) \sim \dot{\tau}^*, \dot{\bar{\tau}}^*$  vanishes in the absence of news

time components BH gravitons

$$\bar{J}_s^u = -\frac{1}{8\pi G} \left\{ \left[ \overbrace{\psi_1^0 + \bar{\psi}_1^0 + \tau^0 \dot{\bar{\tau}}^0 + \bar{\tau}^0 \dot{\tau}^0}^{\text{BH}} \right] f + \left[ \psi_1^0 + \tau^0 \dot{\bar{\tau}}^0 + \frac{1}{2} \dot{\tau}(\tau^0 \bar{\tau}^0) \right] \bar{f} + \left[ \bar{\psi}_1^0 + \bar{\tau}^0 \dot{\bar{\tau}}^0 + \frac{1}{2} \dot{\tau}(\tau^0 \bar{\tau}^0) \right] \bar{f} \right\}$$

$$\Theta_s^u (\delta X) = \frac{1}{8\pi G} \left[ \dot{\tau}^0 \delta \tau^0 + \dot{\bar{\tau}}^0 \delta \bar{\tau}^0 \right] f$$

charges  $Q_s = \int_{S^2, u=cte} \frac{i}{R^2} \frac{\partial \xi \partial \bar{\xi}}{P_s \bar{P}_s} \bar{J}_s^u$

$$(\mathcal{H})_s (\delta X) = \int_{S^2, u=cte} \frac{i}{R^2} \frac{\partial \xi \partial \bar{\xi}}{P_s \bar{P}_s} \Theta_s^u (\delta X)$$

algebra  $\int_{S_1} Q_{S_2} + \Theta_{S_2} [\delta_{S_1} X] = -Q_{[S_1, S_2]}$

(non-)conservation of  $BMS_4$  charges

G.B. & C.Troessert JHEP(2011)

JHEP(2012)

$$\frac{d}{du} Q_s = - \int_{S^2, u=cte} \frac{i}{R^2} \frac{\partial \xi \partial \bar{\xi}}{8\pi G P_s \bar{P}_s} \left[ \dot{\tau}^0 \delta_s \tau^0 + \dot{\bar{\tau}}^0 \delta_s \bar{\tau}^0 \right]$$

fluxes      generalizes Bondi mass loss

non-radiative space-times  
(no news)

$$\tau^{\circ} = \tau^{\circ}(\xi, \bar{\xi}, \cancel{x}) \quad (\Rightarrow \dot{\tau}^{\circ} = 0 = \psi_3^{\circ} = \psi_4^{\circ}, \quad \partial_s[\delta x] = 0)$$

compare "abstract" construction of  $\text{dms}_4^*$

identification of  $w=0$

$$\mathcal{P} = -\frac{1}{2G} (\psi_1^{\circ} + \bar{\psi}_2^{\circ})$$

super-momentum

= Bondi mass aspect

$$\bar{\mathcal{J}} = -\frac{1}{2G} (\psi_1^{\circ} + \tau^{\circ} \mathcal{T} \bar{\tau}^{\circ} + \frac{1}{2} \mathcal{T} (\tau^{\circ} \bar{\tau}^{\circ}))$$

$$\psi_{1\bar{J}}^{\circ}$$

~~super-~~ angular momentum

= Bondi angular momentum

(pre) moment map:  $\mathbb{F}$ . algebra of non-radiative free fields

$\text{dms}_4^*$  representation  $\delta_s$ ,  $[\delta_{s_1}, \delta_{s_2}] = \delta_{[s_1, s_2]}$

aspect

$$\mu: \mathbb{F} \rightarrow \text{dms}_4^*$$

$$\mu \left( -\frac{1}{2G} (\psi_1^{\circ} + \bar{\psi}_2^{\circ}) \right) = \mathcal{P}, \quad \mu \left( -\frac{1}{2G} \psi_{1\bar{J}}^{\circ} \right) = [\bar{\mathcal{J}}], \quad \mu \circ \delta_s = \text{d}\delta_s^* \circ \mu$$

transformation laws at  $u=0$

$$\delta_S (\psi_1^0 + \bar{\psi}_2^0) = (g\hat{f} + \bar{g}\hat{f} + \frac{3}{2}\hat{f}g + \frac{3}{2}\hat{f}\bar{g})(\psi_1^0 + \bar{\psi}_2^0) \quad \checkmark$$

$$\delta_S \psi_1^0 = [g\hat{f} + \bar{g}\hat{f} + 2\hat{f}g + \hat{f}\bar{g}] \psi_1^0 + \frac{1}{2} \mathcal{T} \hat{f} (\psi_2^0 + \bar{\psi}_2^0 + \cancel{\hat{f}^2 f^0} - \cancel{f^2 \bar{f}^0}) \\ + \frac{3}{2} \hat{f} \mathcal{T} (\psi_2^0 + \bar{\psi}_2^0 + \cancel{\hat{f}^2 f^0} - \cancel{f^2 \bar{f}^0})$$

$$\delta_S \psi_{1\bar{f}}^0 = [g\hat{f} + \bar{g}\hat{f} + 2\hat{f}g + \hat{f}\bar{g}] \psi_{1\bar{f}}^0 + \frac{1}{2} \mathcal{T} \hat{f} (\psi_2^0 + \bar{\psi}_2^0) + \frac{3}{2} \hat{f} \mathcal{T} (\psi_2^0 + \bar{\psi}_2^0)$$

$$+ \frac{1}{2} \bar{f} (\mathcal{T} \hat{f} f^0 - \hat{f} \mathcal{T} \hat{f} f^0 + 3\hat{f} \mathcal{T} \bar{f}^0 - 3\bar{f} \mathcal{T} \hat{f}^0 - \frac{3}{2} \mathcal{T} \bar{f}^0) - \frac{1}{2} \hat{f}^3 (\mathcal{T} \bar{f}^0)$$

trivial!

Remark: electric case  $\hat{f}^2 \tau_e^0 = \hat{f}^2 \bar{\tau}_e^0 \Leftrightarrow \tau_e^0 = \hat{f}^2 \chi_e$

Newman-Penrose JMP 1966

$$\delta_S \chi_e = [g\hat{f} + \bar{g}\hat{f} - \frac{1}{2}\hat{f}g - \frac{1}{2}\hat{f}\bar{g}] - \mathcal{T}$$

Strominger et al. 2015-  
Compère et al. 2016

simplified pre-momentum map  $\mu': \mathbb{F}_e \longrightarrow \text{dMSA}^*$

$$\mu'[-\frac{1}{2G}(\psi_1^0 + \bar{\psi}_2^0)] = \mathcal{T}, \quad \mu'[-\frac{1}{2G}\psi_1^0] = [\bar{g}], \quad \mu' \circ \delta_S = \text{ad}_S^* \circ \mu'$$

## Perspectives

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1) on punctured plane  $\int^3 \tilde{y} \neq 0$

$\exists$  super-rotations & super-singular momentum

$$K_{S_1, S_2} = \text{Res}_z \text{Res}_{\bar{z}} [r^0 f_1 \delta^3 h_2 - (12)^+ \text{c.c.}]$$

$\exists$  field-dependent central extension & associated Souriau cocycle

GB & Troessaert JHEP 2016

GB JHEP 2017

mapping from plane to cylinder to make  $r^0$  non-zero  
for Kerr

2) coadjoint repres. of generalized BMS<sub>4</sub> Campiglia & Laddha Phys. Rev. 2014

$$\text{Diff}(S^2) \times C(S^2) \quad \text{on } S^2 \quad \text{drop} \quad f \tilde{y} = 0 = \bar{f} y \quad f^3 y = 0 = \bar{f}^3 \bar{y}$$

but also equivalence relations

$$\bar{y} \sim \tilde{y} + \bar{f} \bar{y}, \quad \bar{f} \sim \tilde{f} + f^3 M$$

related groups  
Donnelly et al. 2020

simply expand everything in spin-weighted spherical harmonics

### 3) Classify coadjoint orbits

symplectic manifolds can be quantized  $\rightarrow$  relation to VIRREPS of  $SU(4)$

Write geometric actions

Alekseev, Faddeev, Shatashvili J.Geom. Phys. 1988, Nucl. Phys. 1989

$$S = \int \text{Tr} (g^{-1})_\mu g \tilde{g}^\mu J^\mu g) d^4x \rightarrow S = \int \left( \langle n_0^*, g^{-1} \frac{dg}{dt} \rangle - H \right) dt$$

for 3d gravity  $\Leftrightarrow$  to actions constructed from

$CS \rightarrow W + \omega$  Elitzur et al. Nucl. Phys. 1989

GB, Groutzler, Salgado CQG 2018

Coussaert, Henneaux, Teitelboim CQG 1995

no killing metric needed. correct global symmetries

Effective actions for Goldstone bosons

4) Complete pre-momentum map to bona fide one  
connection to spatial infinity Henneaux & Troessaert JHEP 2018

Torre CQG 1986

Oliveri & Speziale 2019

Wieland 2020

5) Study interactions of this group theory sector with  
radiative Dof at  $J^+$

Ashtekar & Streubel Proc. Roy. Soc. 1981

Ashtekar (1984)

6) Krichever-Novikov algebras for more than 2 punctures?