Higher-order couplings of three-dimensional O(n)-symmetric ϕ^4 theory: multiloop renormalization-group analysis

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Presentation plan

Devision of Quantum Mechanics

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1. Critical phenomena and field theory
2. Research goals
3. Renormalization group expansions of $\mathbf{g_8}$ and $\mathbf{g_{10}}$
4. Handling of expansions obtained, numerical estimates of couplings and their universal ratios $\mathbf{R_8}$ and $\mathbf{R_{10}}$:
   1. Borel based resummation
   2. Pseudo-epsilon expansion as resummation approach
5. Conclusion

Generalized Heisenberg model in the theory of critical behavior

This model describes critical phenomena in a broad class of materials such as:

1. $\mathbf{n = 1}$ - easy-axis ferromagnets, simple fluids and binary mixtures;
2. $\mathbf{n = 2}$ - easy-plane ferromagnets, some of the superconductors and superfluid helium-4;
3. $\mathbf{n = 3}$ - Heisenberg ferromagnets;
4. $\mathbf{n = 4}$ - quark-gluon plasma in some quantum chromodynamics models;
5. limiting regimes of the critical behavior of superfluid liquids with triplet pairing: helium-3 ($\mathbf{n = 18}$) and matter of neutron stars ($\mathbf{n = 10}$).

Free energy, nonlinear susceptibilities

Critical behavior of system can be described by means of effective coupling constant $\mathbf{g_{2k}}$ and their universal ratios $\mathbf{R_{2k} = g_{2k}/g_4^{(k-1)}}$.

Free energy:

$$\mathbf{F(z,m) - F(0,m) = \frac{m^3}{g_4}\left(\frac{z^2}{2} + z^4 + R_6 z^6 + R_8 z^8 + R_{10} z^{10} \ldots\right)}, \tag{1}$$

Nonlinear susceptibilities:

$$\chi_4 = \left.\frac{\partial^3 \mathbf{M}}{\partial \mathbf{H}^3}\right|_{\mathbf{H}=0} = -\mathbf{24}\frac{\chi^2}{\mathbf{m^3}}\mathbf{g_4}, \tag{2}$$

$$\chi_6 = \left.\frac{\partial^5 \mathbf{M}}{\partial \mathbf{H}^5}\right|_{\mathbf{H}=0} = -\mathbf{6!}\frac{\chi^3 \mathbf{g_4^2}}{\mathbf{m^6}}(\mathbf{R_6 - 8}), \tag{3}$$

Nonlinear susceptibilities:

$$\chi_8 = \left.\frac{\partial^7 \mathbf{M}}{\partial \mathbf{H}^7}\right|_{\mathbf{H}=0} = -8!\,\frac{\chi^4 \mathbf{g}_4^3}{\mathbf{m}^9}(\mathbf{R}_8 - 24\mathbf{R}_6 + 96), \tag{4}$$

$$\chi_{10} = \left.\frac{\partial^9 \mathbf{M}}{\partial \mathbf{H}^9}\right|_{\mathbf{H}=0} = -10!\,\frac{\chi^5 \mathbf{g}_4^4}{\mathbf{m}^{12}}(\mathbf{R}_{10} - 32\mathbf{R}_8 - 18\mathbf{R}_6^2 + 528\mathbf{R}_6 - 1408) \tag{5}$$

Here $\mathbf{z} = \mathbf{M}\sqrt{\mathbf{g_4}/\mathbf{m}^{1+\eta}}$ - dimensionless magnetization, renormalized mass, $\mathbf{m} \sim (\mathbf{T} - \mathbf{T_c})^{\nu}$, χ - linear susceptibilities, and χ_4, χ_6, χ_8 и χ_{10} - nonlinear susceptibilities of forth, sixth, eighth, and tenth orders.

Results were obtained earlier

For an arbitrary value of the order parameter dimensionality, the coupling constants $\mathbf{g_6}$ and $\mathbf{g_8}$ were obtained earlier in four and three-loop approximations*.

In particular, in the case of $\mathbf{n = 1}$ (Ising model), the calculation of the universal critical values $\mathbf{g_4}$, $\mathbf{g_6}$, and $\mathbf{R_6}$ within the field-theoretical renormalization group(RG) approach using different resummation techniques leads to a spread of numerical results for $\mathbf{g_6}$ less than 0.5 $\%$ **.

*A. I. Sokolov, E. V. Orlov, V. A. Ul'kov, and S. S. Kashtanov, Phys. Rev. E 60, 1344 (1999)
A. I. Sokolov, E. V. Orlov, V. A. Ul'kov, Phys. Lett. A **227 (1997) 255.
R. Guida, J. Zinn-Justin, Nucl. Phys. B **489 (1997) 626.

What was found?

In this talk one will be presented:

- renormalization group expansions of record length for an arbitrary $\mathbf{n}$ - four-loop expansion for $\mathbf{g_8}$ and three-loop expansion for $\mathbf{g_{10}}$;
- numerical estimates of universal ratios obtained with the help of various resummation techniques;
- comparative analysis with results obtained in the special case - $\mathbf{n = 1}$ *.

*R. Guida, J. Zinn-Justin, Nucl. Phys. B **489** (1997) 626.

The model used

The three-dimensional n-vector model is described by the following Hamiltonian:

$$\mathbf{H} = \int \mathbf{d^3x} \left[\frac{\mathbf{1}}{\mathbf{2}} (\mathbf{m_0^2} \varphi_\alpha^{\mathbf{2}} + (\nabla \varphi_\alpha^{\mathbf{2}}) + \frac{\lambda}{\mathbf{24}} (\varphi_\alpha^{\mathbf{2}})^{\mathbf{2}} \right], \qquad (6)$$

where φ_α - $\mathbf{n}$-component field, square of bare mass - $\mathbf{m_0^2}$ - is proportional to $\mathbf{T} - \mathbf{T_c^{(0)}}$, $\mathbf{T_c^{(0)}}$ - critical temperature in mean-field approximation.

Renormalization

Taking into account the fluctuations leads to mass renormalization $\mathbf{m_0^2} \to \mathbf{m^2}$, field $\varphi \to \varphi_\mathbf{R}$, and coupling constant $\lambda \to \mathbf{mg_4}$, and also to the appearance of higher-order terms in the expansion for the free energy in powers of the magnetization $\mathbf{M}$:

$$\mathbf{F(M,m)} = \mathbf{F(0,m)} + \sum_{\mathbf{k=1}}^{\infty} \mathbf{\Gamma_{2k} M^{2k}}. \tag{7}$$

Coefficients of expansion $\mathbf{\Gamma_{2k}}$ comprise complete vertices with $2k$ external (truncated) lines which are linked by simple relations with $2n$-point 1-irreducible correlation functions $\mathbf{G_{2k}(q_1, q_2, \ldots, q_{2n-1})}$ at zero momenta.

Renormalization

In the critical region, each of the vertices has its own scale dimensionality:

$$\mathbf{\Gamma_{2k} = g_{2k} m^{3-k(1+\eta)}}, \tag{8}$$

where η - Fisher exponent, a $\mathbf{g_{2k}}$ - some constants. Expanding the free energy in powers of the dimensionless magnetization $\mathbf{z = M\sqrt{g_4/m^{(1+\eta)}}}$, we obtain the following expression:

$$\mathbf{F(z,m) - F(0,m) = \frac{m^3}{g_4}\left(\frac{z^2}{2} + z^4 + R_6 z^6 + R_8 z^8 + R_{10} z^{10} \ldots\right)}. \tag{9}$$

Renormalization

Under $\mathbf{T} \to \mathbf{T_c}$ ratios of couplings $\mathbf{R_{2k}}$ take some universal values, which together determine the form of the equation of state in the region of strong fluctuations.

How to get RG expansion?

$\mathbf{g_8}$: Since in three dimensions only $\lambda\varphi^{\mathbf{4}}$ type interaction is relevant in the RG only four-point bare vertices will appear in the diagrams of perturbative series. Having obtained the expansion for $\mathbf{\Gamma_8}$ in powers of λ, we can renormalize it by expressing λ in terms of $\mathbf{g_4}$ using the well-known relation:

$$\lambda = \mathbf{m Z_4 Z^{-2} g_4}, \tag{10}$$

where $\mathbf{Z_4}$ и $\mathbf{Z}$ - the renormalization constants of the interaction λ and the field φ: $\varphi = \sqrt{\mathbf{Z}}\varphi_{\mathbf{R}}$.

g_8: RG expansion in the four-loop approximation

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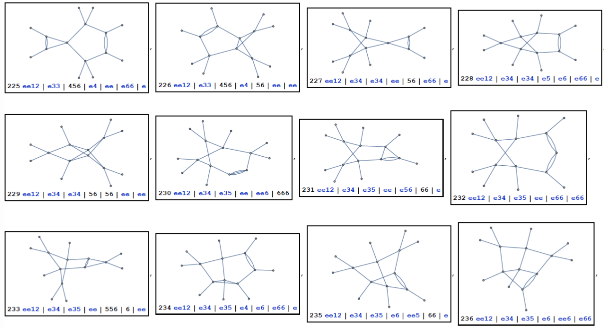
In the four-loop approximation the expression for g_8 is given by 310 (1 + 5 + 36 + 268) Feynman diagrams. Some - four-loop - of them:



$\mathbf{g_8}$: RG expansion in the four-loop approximation

RG expansion:

$$\mathbf{g_8} = -\frac{\mathbf{81}}{\mathbf{2}\pi}\mathbf{g_4^4}\left(\frac{\mathbf{n+80}}{\mathbf{81}} - \mathbf{g_4}\frac{(\mathbf{81n^2+7114n+134960})}{\mathbf{13122}\pi} + \right.$$
$$+\mathbf{g_4^2}\left(\mathbf{0.00943497n^2+0.609413n+7.15615}\right) +$$
$$\left. + \underbrace{\mathbf{g_4^3}\left(-\mathbf{0.000131n^3-0.047038n^2-1.97177n-16.5648}\right)}_{\text{new term - sum of 268 diagrams}}\right), \tag{11}$$

$$\mathbf{g_4} = \frac{\mathbf{2}\pi}{\mathbf{n+8}}\mathbf{g}, \tag{12}$$

where $\mathbf{g}$ - effective coupling constant, which weakly depends on order parameter dimensionality $\mathbf{n}$.

g_{10}: RG expansion in the three-loop approximation

RG expansion:

$$\mathbf{g_{10}} = \frac{\mathbf{243}}{\pi}\mathbf{g_4^5}\left(\frac{\mathbf{242+n}}{\mathbf{243}} + \mathbf{g_4}\frac{\left(\mathbf{-380150 - 13429n - 81n^2}\right)}{\mathbf{19683}\pi} + \right. \\ \left. + \mathbf{g_4^2}\left(\mathbf{21.741482 + 1.421197n + 0.021392n^2 + 0.000104n^3}\right)\right) \tag{13}$$

Comparison with existing results for the Ising model($\mathbf{n = 1}$):

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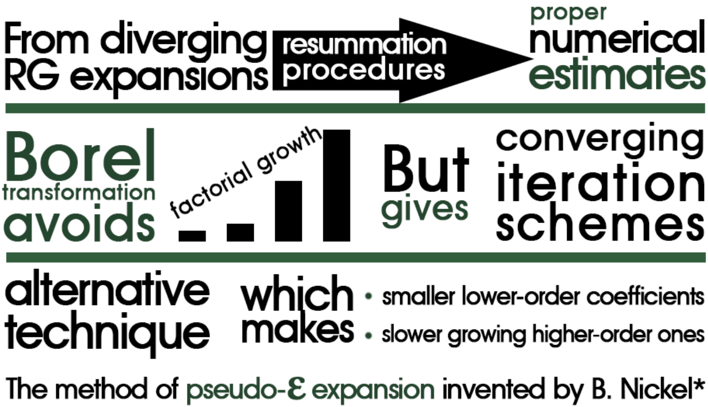
Three-loop (for $\mathbf{g_{10}}$) and four-loop (for $\mathbf{g_8}$) terms under $\mathbf{n = 1}$:

	g_8	$\mathbf{g_{10}}$
Our result	239.5735887	1793.279821
R. Guida, J. Zinn-Justin*	239.5735884	1793.279824

*R. Guida, J. Zinn-Justin, Nucl. Phys. B **489** (1997) 626.

Resummation



*Ref. 19 in J. C. Le Guillou, J. Zinn-Justin, Phys. Rev. B **21** (1980) 3976.

Pade-Borel resummation

This approach uses Borel transformation of initial divergent series:

$$\mathbf{f(x)=\sum_{i=0}^{\infty} c_i x^i = \int\limits_0^{\infty} e^{-t} t^b F(xt) dt}, \quad \mathbf{F(y)=\sum_{i=0}^{\infty} \frac{c_i}{(i+b)!} y^i}, \tag{14}$$

with subsequent analytic continuation in the form of Pade approximants. How to choose a parameter value $\mathbf{b}$?

Pade-Borel resummation

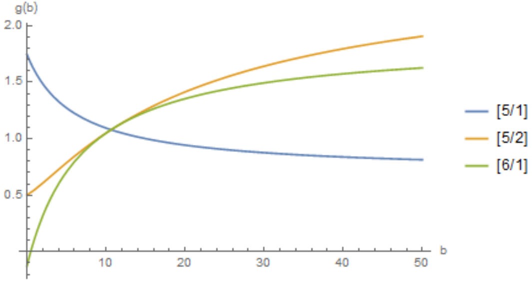


Рис. : $\mathbf{g_8 = 1.07926,\ n = 1,\ b = 10.6401}$

One choose the value $\mathbf{b}$ so, that distance between the most stable approximants - [5/1],[5/2] и [6/1] - would be minimal.

Pade-Borel resummation

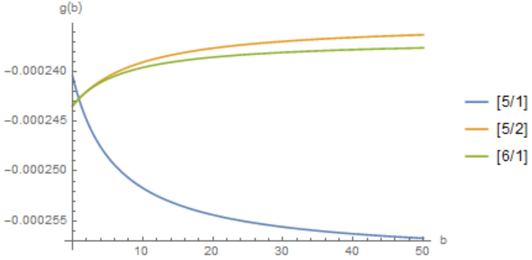


Рис. : $\mathbf{g_8 = -0.0002171,\ n = 64,\ b = 0.67515}$

It is easy to see, that with increasing of $\mathbf{n}$ the dependence of the approximants on the fitting parameter $\mathbf{b}$ decreases.

Results of PBL resummation of g_8

n	g_8	b
0	2.1702	10.958
1	1.0793	10.640
2	0.5326	10.488
3	0.2543	10.433
4	0.1115	10.433
5	0.0387	10.461
6	0.0022	10.499
7	−0.0153	10.533
8	−0.0227	10.554
9	−0.0250	10.557
10	−0.0247	10.539
16	−0.0130	9.9224
24	−0.0049	8.1496
32	−0.0022	6.1182
48	−0.00059	2.8152
64	−0.00022	0.6749

Alternative way to obtain numerical estimates is pseudo-ϵ expansion approach.

Pseudo-ϵ expansion for fixed point coordinate

We take the well-known six-loop RG expansion of the β-function* for the 3D $\mathbf{n}$-vector model and introduce the formal small parameter τ before the linear term:

$$-\beta(\mathbf{g}) = \tau\mathbf{g} - \mathbf{g^2} + \dots .$$

Then one iteratively solve the equation for the coordinate of a fixed point:

$$\beta(\mathbf{g}^*) = \mathbf{0} \quad \Rightarrow \mathbf{g}^* = \mathbf{a_1}\tau + \mathbf{a_2}\tau^{\mathbf{2}} + \dots .$$

Thus, in six-loop approximation we have (we show only the first couple of terms, because of the cumbersomeness of the whole expression):

$$\mathbf{g} = \tau + \tau^{\mathbf{2}} \frac{(\mathbf{28.14814815 + 6.07407408n})}{(\mathbf{8+n})^{\mathbf{2}}} + \dots$$

*S.A. Antonenko, A.I. Sokolov. Phys. Rev. E51, 3, 1894 (1995).

Pseudo-ϵ expansion for g_8

$$\mathbf{g_8} = -\frac{(8(80+n)\pi^3)}{(8+n)^4}\tau^4 + \frac{(1072067 + 77514.16n + 17743.25n^2 + 248.0502n^3)}{(8+n)^6}\tau^5 +$$
$$+\frac{1}{(8+n)^8}(8711449 + 1592297 * 10n + 1715307n^2 + 197852.9n^3 +$$
$$+1387.955n^4)\tau^6 + \frac{1}{(8+n)^{10}}(4.90159 * 10^9 + 2.16335 * 10^9 n +$$
$$+4.46135 * 10^8 n^2 + 2.88041 * 10^7 n^3 + 1204930n^4 - 37159.0n^5 -$$
$$-866.770n^6)\tau^7$$

This series as in previous cases has to be resummed, after that one can equate $\tau = 1$. Actually, the numerical estimates for the universal ratio $\mathbf{R_8} = \mathbf{g_8}/\mathbf{g_4^3}$ are usually given, since it enters the expression for free energy.

Summary table for $\mathbf{R_8}$

n	4-loop PBL	4-loop PϵE	3-loop PBL [2]	ϵ-exp.[1]	LC
0	1.548	1.382		1.1(2)	
1	1.109	0.968	0.856	0.94(14)	0.871(14)[3]
			0.857(86)[4]	0.78(5)[4]	0.79(4)[5]
2	0.770	0.646	0.563	0.71(16)	0.494(34)[6]
3	0.506	0.397	0.334	0.33(10)	0.21(7)[7]
4	0.299	0.206	0.15	0.065(80)	0.07(14)[8]
5	0.137	0.058	−0.3(9)[9]	−0.1(2)[9]	
6	0.010	−0.056	−0.09	−0.2(1)[9]	
7	−0.090	−0.144			
8	−0.169	−0.212	−0.25	−0.405(31)	
10	−0.280	−0.306			
16	−0.412	−0.412	−0.44	−0.528(14)	
24	−0.424	−0.413			
32	−0.394	−0.380	−0.42	−0.425(7)	
			−0.45(7)[9]	−0.427(3)[9]	
48	−0.324	−0.311	−0.35	−0.322(2)	
64	−0.270	−0.259	−0.29(3)[9]	−0.269(3)[9]	

1. A. Pelissetto, E. Vicari, Nucl. Phys. B **575** (2000) 579; 2. A. I. Sokolov, E. V. Orlov, V. A. Ul'kov, and S. S. Kashtanov, Phys. Rev. E **60 (1999) 1344; 3 P. Butera, M. Pernici, Phys. Rev. B 83 (2011) 054433; 4. R. Guida, J. Zinn-Justin, Nucl. Phys. B 489 (1997) 626; 5. M. Campostrini, A. Pelissetto, P. Rossi, E. Vicari, Phys. Rev. E 65 (2002) 066127; 6. M. Campostrini, M. Hasenbusch, A. Pelissetto, E. Vicari, J. Phys. A 34 (2001) 2923; 7. M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi, E. Vicari, Phys. Rev. B 65 (2002) 144520; 8. F. P. Toldin, A. Pelissetto, and E. Vicari, J. High Energy Phys. 07 (2003) 029; 9. A. Butti and F. P. Toldin, Nucl. Phys. B 704 (2005) 527.**

Conclusion

- the four-loop contribution to the RG expansion essentially shifted the numerical estimates for $\mathbf{R_8}$ under physical values of the order parameter dimensionality $\mathbf{n}$;
- under these $\mathbf{n}$ the numerical estimates for $\mathbf{R_8}$ still notably depend on the resummation procedure;
- the three-loop RG expansion for $\mathbf{R_{10}}$ found, unfortunately, does not allow us to obtain any stable numerical estimates for this ratio.

From all the above, it can be concluded, that it is necessary to continue the development of methods for calculation of multi-loop contributions to the expansions for effective coupling constants, which will allow us to obtain the reliable numerical results.

Thank you for your attention!