Renormalization of gauge theories in the background-field approach

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Task: to prove that counterterms are covariant local functionals of the original gauge field

Gauge-breaking and ghost terms – counterterms to them?

Preservation of the BRST structure and counterterms covariance by
1) *The choice of background covariant gauge conditions*
and
2) *gauge field reparametrization*

(DeWitt, Tuytin-Voronov, Batalin-Vilkovisky, Kallosh, Arefieva-Faddeev-Slavnov, Abbot, Henneaux et al)

40+ years old topic! So what is new here?

D.Blas, M.Herrero-Valeya, S.Sibiryakov, C.Steinwachs & A.B.
Phys. Rev. D 93, 064022 (2016), arXiv:1512.02250;
arXiv:1705.03480; PRL 119,211301 (2017), arXiv:1706.06809

Background field extension of the BRST operator

Inclusion of generating functional sources into the gauge fermion

Invariance of gauge fields counterterms via decoupling of the background field

Quantum corrected gauge fermion is a generating functional of the field reparameterization

No power counting or use of field dimensionalities

Extension to Lorentz symmetry violating theories

Extension to (nonrenormalizable) effective field theories

BRST formalism

Gauge theory:

$$\varphi = \varphi^a, \quad S = S[\varphi], \quad \frac{\delta S}{\delta \varphi^a} R^a_\alpha = 0$$

Generators of gauge transformations:

$$R^a_\alpha = R^a_\alpha(\varphi), \quad \delta_\epsilon \varphi^a = R^a_\alpha \epsilon^\alpha,$$

$$R^a_\alpha \frac{\delta R^b_\beta}{\delta \varphi^a} - R^a_\beta \frac{\delta R^b_\alpha}{\delta \varphi^a} = C^\gamma_{\alpha\beta} R^b_\gamma$$

structure constants

DeWitt summation rule: $a = (A, x), \quad F^a \Psi_a \equiv \int dx\, F^A(x) \Psi_A(x)$

Feynman-DeWitt-Faddeev-Popov functional integral

$$e^{-W[\,J\,]}=\int d\varphi\, e^{-S[\,\varphi\,]-\underbrace{\frac{1}{2}\chi^{\alpha}O_{\alpha\beta}\chi^{\beta}}_{\text{gauge-breaking term}}-J\varphi}\,\underbrace{\big(\mathrm{det}O_{\alpha\beta}\big)^{1/2}}_{\text{measure}}\,\mathrm{det}\underbrace{\left(\frac{\delta\chi^{\alpha}}{\delta\varphi^{a}}R^{a}_{\beta}\right)}_{\text{Faddeev-Popov operator}}$$

$$S[\,\varphi\,]\to\Sigma[\,\Phi\,]=S[\,\varphi\,]+b_{\alpha}\chi^{\alpha}(\varphi)-\frac{1}{2}O^{\alpha\beta}\,b_{\alpha}b_{\beta}-\bar{\omega}_{\alpha}\left(\frac{\delta\chi^{\alpha}}{\delta\varphi^{a}}R^{a}_{\beta}\right)\omega^{\beta}$$

$$\varphi^{a}\to\Phi=\varphi^{a},\omega^{\alpha},\bar{\omega}_{\alpha},b_{\alpha}$$

BRST functional integral

$$e^{-W[\,\mathcal{J}\,]}=\int d\Phi\, e^{-\Sigma[\,\Phi\,]-\mathcal{J}\Phi}$$

$$\Sigma[\Phi] = S[\varphi] + s\Psi[\Phi]$$

BRST action

$$s = (s\Phi)\frac{\delta}{\delta\Phi}, \qquad s^2 = 0$$

nilpotent BRST operator

BRST transformations of Φ

$$s\Phi: \quad s\varphi^a = R^a_\alpha(\varphi)\,\omega^\alpha\,, \quad s\omega^\alpha = \frac{1}{2}C^\alpha_{\beta\gamma}\,\omega^\beta\omega^\gamma,$$

$$s\bar{\omega}_\alpha = b_\alpha, \quad sb_\alpha = 0\,.$$

Gauge fermion

(correlators of physical observable are independent of Ψ)

$$\Psi[\Phi] = \bar{\omega}_\alpha\Big(\chi^\alpha(\varphi) - \frac{1}{2}O^{\alpha\beta}\,b_\beta\Big)$$

gauge conditions

gauge-fixing matrix

BRST invariance

$$s\Sigma[\Phi] = 0$$

Assumptions on the class of theories

generators: linear $\delta_\varepsilon \varphi^a = R^a{}_\alpha(\varphi)\,\varepsilon^\alpha, \quad R^a{}_\alpha(\varphi) = P^a{}_\alpha + R^a{}_{b\alpha}\varphi^b$

closed algebra $\left[\delta_\varepsilon, \delta_\eta\right]\varphi^a = \delta_\varsigma \varphi^a \quad \varsigma^\alpha = C^\alpha{}_{\beta\gamma}\varepsilon^\beta \eta^\gamma$

ireducible $R^a{}_\alpha \varepsilon^\alpha = 0 \;\Rightarrow\; \varepsilon^\alpha = 0$

Examples:

YM: $\delta_\varepsilon A^i_\mu = f^{ijk} A^j_\mu \varepsilon^k + \partial_\mu \varepsilon^i$

GR: $\delta_\varepsilon g_{\mu\nu} = \varepsilon^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda}\partial_\nu \varepsilon^\lambda + g_{\nu\lambda}\partial_\mu \varepsilon^\lambda$

Also higher-derivative gravity, also non-relativistic (Lifshitz) theories

Counterexample:

Supergravity (the algebra does not close off-shell)

Existence of gauge invariant regularization (no anomalies) – very subtle!

Background gauge fixing

background field

$$\chi^\alpha(\varphi) \to \chi^\alpha(\varphi, \phi) = \chi^\alpha_a(\phi)(\varphi^a - \phi^a)$$

background gauge transformations (BGT)

$$\delta_\varepsilon \varphi^a = R^a{}_\alpha(\varphi)\,\varepsilon^\alpha\ , \qquad \delta_\varepsilon \phi^a = R^a{}_\alpha(\phi)\,\varepsilon^\alpha$$

same

Linear representation of the gauge group:

$$\delta_\varepsilon(\varphi^a - \phi^a) = R^a{}_{b\alpha}(\varphi^b - \phi^b)\varepsilon^\alpha$$

fundamental representation

$$\delta_\varepsilon \chi^\alpha \equiv \frac{\delta \chi^\alpha}{\delta \varphi^a}\delta_\varepsilon \varphi^a + \frac{\delta \chi^\alpha}{\delta \phi^a}\delta_\varepsilon \phi^a = -C^\alpha{}_{\beta\gamma}\chi^\beta \varepsilon^\gamma$$

adjoint representation

Modification of BRST operator

$$\boldsymbol{s} \to \boldsymbol{Q} = \boldsymbol{s} + \Omega^a \frac{\delta}{\delta \phi^a}$$

anticommuting auxiliary field – controls dependence on background ϕ

$$\Sigma \to S[\varphi] + \boldsymbol{Q}\Psi, \qquad \boldsymbol{Q}\Sigma = 0$$

Introduction of sources into gauge fermion

$$\mathcal{J} = \underbrace{J_a,\ \bar{\xi}_\alpha,\ \xi^\alpha,\ y^\alpha}_{\text{sources for } \Phi = \varphi^a, \omega^\alpha, \bar{\omega}_\alpha, b_\alpha},\ \underbrace{\gamma_a,\ \zeta_\alpha}_{\text{sources for } \boldsymbol{s}\varphi^a, \boldsymbol{s}\omega^\alpha},\ \Omega^a$$

background source

sources for $\Phi = \varphi^a, \omega^\alpha, \bar{\omega}_\alpha, b_\alpha$

sources for $\boldsymbol{s}\varphi^a, \boldsymbol{s}\omega^\alpha$

$$e^{-\frac{1}{\hbar}W[\mathcal{J}]} = \int d\Phi\, e^{-\frac{1}{\hbar}[\Sigma + J_a(\varphi^a - \phi^a) + \bar{\xi}_\alpha \omega^\alpha + \xi^\alpha \bar{\omega}_\alpha + y^\alpha b_\alpha]}$$

$$\Sigma[\Phi, \mathcal{J}] = S[\varphi] + \boldsymbol{Q}\,\Psi[\Phi, \mathcal{J}]$$

$$\Psi[\Phi, \phi, \gamma, \zeta] = \bar{\omega}_\alpha \Big(\chi^\alpha_a(\phi)(\varphi^a - \phi^a) - \frac{1}{2} O^{\alpha\beta}(\phi)\, b_\beta\Big)$$
$$- \gamma_a(\varphi^a - \phi^a) + \zeta_\alpha \omega^\alpha$$

Obviously $\quad \varphi^a - \phi^a = -\dfrac{\delta\Psi}{\delta\gamma_a}, \qquad \omega^\alpha = \dfrac{\delta\Psi}{\delta\zeta_\alpha}$

Introduction of sources into BRST charge

Extended BRS operator

$$\boldsymbol{Q} \to \boldsymbol{Q}_{\mathrm{ext}} = \boldsymbol{s} + \Omega\frac{\delta}{\delta\phi} - J\frac{\delta}{\delta\gamma} + \bar{\xi}\frac{\delta}{\delta\zeta} + \xi\frac{\delta}{\delta y}, \qquad \boldsymbol{Q}_{\mathrm{ext}}^2 = 0$$

$$\xi\bar{\omega} + yb = \left(\boldsymbol{s} + \xi\frac{\delta}{\delta y}\right) y\bar{\omega} = \boldsymbol{Q}_{\mathrm{ext}}\,(y\bar{\omega})$$

Extended gauge fermion

$$\Psi \to \boldsymbol{\Psi}_{\mathrm{ext}} \equiv \Psi + y\bar{\omega}$$

$$\Sigma \to \Sigma_{\mathrm{ext}} = \Sigma - J\frac{\delta\Psi}{\delta\gamma} + \bar{\xi}\frac{\delta\Psi}{\delta\zeta} + \xi\bar{\omega} + yb = S + \boldsymbol{Q}_{\mathrm{ext}}\boldsymbol{\Psi}_{\mathrm{ext}}$$

$$\boxed{e^{-W/\hbar} = \int d\Phi\, e^{-(S+\boldsymbol{Q}_{\mathrm{ext}}\boldsymbol{\Psi}_{\mathrm{ext}})/\hbar}}$$

Full set of symmetries

BRST symmetry $$Q_{\mathrm{ext}}\Sigma_{\mathrm{ext}} = 0$$

Add background gauge transformation of other fields and sources:

$$\delta_\varepsilon \gamma_a = -\gamma_b R^b{}_{a\alpha}\varepsilon^\alpha \ , \qquad \delta_\varepsilon \omega^\alpha = -C^\alpha{}_{\beta\gamma}\,\omega^\beta \varepsilon^\gamma$$

$$\delta_\varepsilon \zeta_\alpha = \zeta_\beta C^\beta{}_{\alpha\gamma}\varepsilon^\gamma, \qquad \delta_\varepsilon \Omega^\alpha = R^a{}_{b\alpha}\Omega^b\varepsilon^\alpha$$

BGT symmetry $$\delta_\varepsilon \Psi_{\mathrm{ext}} = 0, \quad \delta_\varepsilon \Sigma_{\mathrm{ext}} = 0$$

U(1) symmetry with ghost numbers:

$$\mathrm{gh}(\varphi) = \mathrm{gh}(\phi) = \mathrm{gh}(b) = 0 \ , \qquad \mathrm{gh}(\omega) = \mathrm{gh}(\Omega) = +1 \ ,$$

$$\mathrm{gh}(\bar\omega) = \mathrm{gh}(\gamma) = -1 \ , \qquad \mathrm{gh}(\zeta) = -2$$

Renormalization at a glance

generating functional $$W[J] = -\hbar \log \int d\Phi \exp\left[-\frac{1}{\hbar}(S[\Phi] + J\Phi)\right]$$

mean fields, effective action $$\langle\Phi\rangle = \frac{\delta W}{\delta J}\,, \qquad \Gamma[\langle\Phi\rangle] = W - J\langle\Phi\rangle$$

subtraction $$S_{\rm ren} = S_0[\Phi] - \hbar\,\Gamma^{1-\rm loop}_{\rm div}[\Phi] - \hbar^2\Gamma^{2-\rm loop}_{\rm div}[\Phi] - \ldots$$

local

Apply it to a gauge theory: effective action renormalized in (L-1)-th order

$$\mathit{\Gamma}_{L-1} = \mathit{\Sigma}_0 + \sum_{l=1}^{\infty} \hbar^l \mathit{\Gamma}^{(l)}_{L-1}, \qquad \mathit{\Gamma}^{(L)}_{L-1,\infty} \equiv \mathit{\Gamma}_{L,\infty}[\langle\mathit{\Phi}\rangle, \phi, \gamma, \zeta, \mathit{\Omega}]$$

-- local divergence

L-th order renormalization:

quantum field

$$\mathit{\Sigma}_L[\mathit{\Phi}, \phi, \gamma, \zeta, \mathit{\Omega}] = \mathit{\Sigma}_{L-1} - \hbar^L \mathit{\Gamma}_{L,\infty}[\mathit{\Phi}, \phi, \gamma, \zeta, \mathit{\Omega}]$$

Main result

Renormalized generating functional:

renormalized fields

$$e^{-W/\hbar} = \int d\Phi e^{-\left(S+\boldsymbol{Q}\Psi+J_a(\tilde{\varphi}^a-\phi^a)+\bar{\xi}_\alpha\tilde{\omega}^\alpha+\xi^\alpha\bar{\omega}_\alpha+y^\alpha b_\alpha\right)/\hbar}$$

Renormalization of the action and gauge fermion

$$S[\varphi] = S_0[\varphi] - \sum_{l=1}^{\infty} \hbar^l \boldsymbol{S}_l[\varphi]$$

local invariants of the original gauge field

ghost fields and sources dependent

$$\Psi = \Psi_0 - \sum_{l=1}^{\infty} \hbar^l \boldsymbol{\Upsilon}_l[\Phi, \mathcal{J}]$$

$$\Psi_0 = -\hat{\gamma}_a(\varphi^a - \phi^a) + \zeta_\alpha\omega^\alpha - \frac{1}{2}\bar{\omega}_\alpha O^{\alpha\beta}(\phi)b_\beta + y_\alpha\bar{\omega}_\alpha$$

$$\hat{\gamma}_a = \gamma_a - \bar{\omega}_\alpha\chi_a^\alpha$$

Gauge fermion is a generating function of the field redefinition

$$\tilde{\varphi}(\Phi, \mathcal{J}) - \phi = -\frac{\delta\Psi[\Phi, \mathcal{J}]}{\delta\gamma}\,, \quad \tilde{\omega} = \frac{\delta\Psi[\Phi, \mathcal{J}]}{\delta\zeta}$$

$$\Psi[\,\Phi, \mathcal{J}\,] = \Psi[\,\varphi, \omega, \phi, \gamma, \zeta, \Omega\,]$$

$$\Downarrow$$

Local generally nonlinear reparametrization of quantum fields to composite operators including external sources

$$\tilde{\varphi}^a = \tilde{\varphi}^a(\varphi, \omega, \phi, \gamma, \zeta, \Omega) \qquad \tilde{\omega}^\alpha = \tilde{\omega}^\alpha(\varphi, \omega, \phi, \gamma, \zeta, \Omega)$$

Applies to (non-renormalizable) EFT -- nonlinear dependence on sources

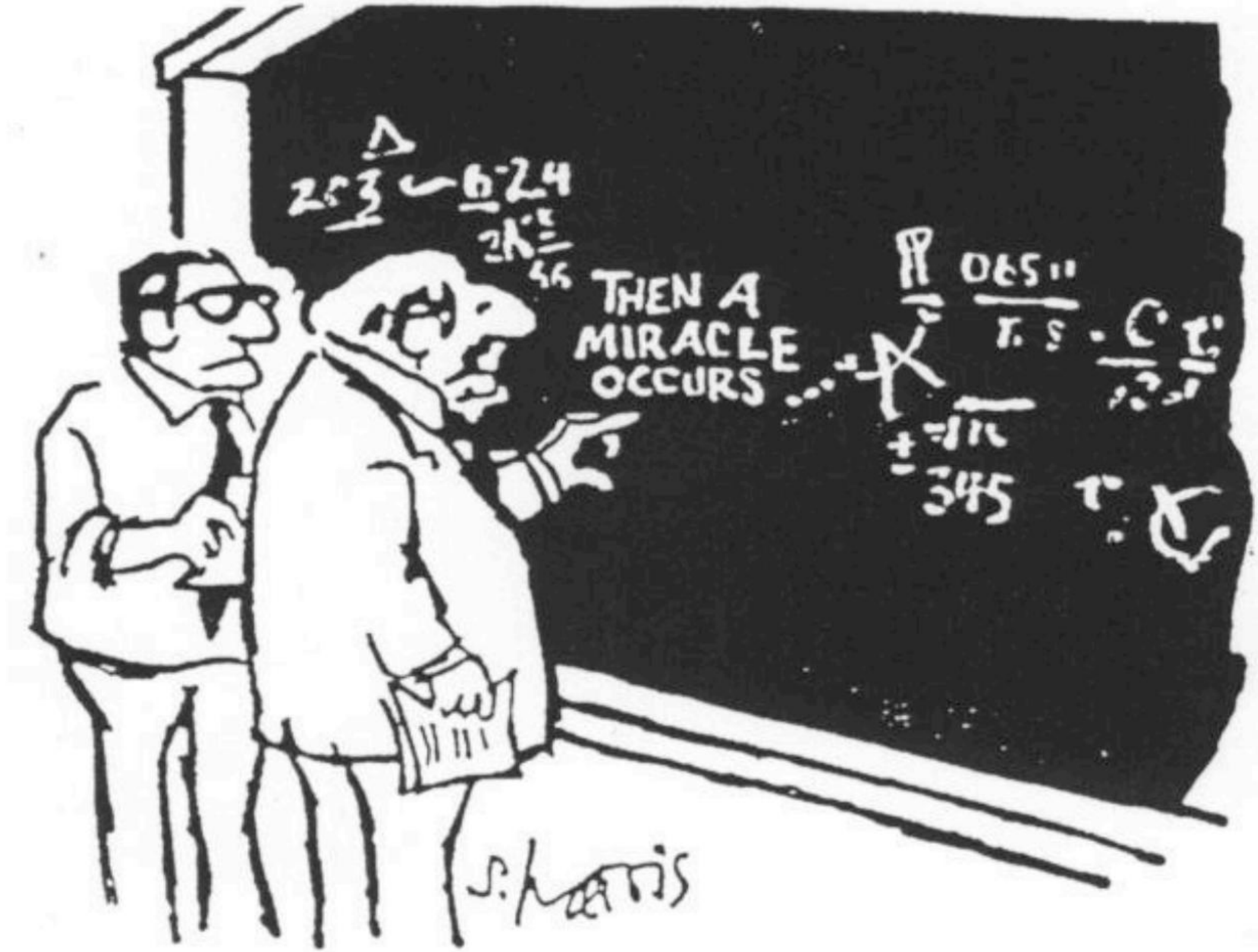
For renormalizable theories:

$$\hat{\Psi}_L = -\hat{\gamma}_a U_L{}^a(\varphi, \phi) + \zeta_\alpha \omega^\beta V_L{}^\alpha_\beta(\varphi, \phi)$$

linear in γ and ζ

$$\tilde{\varphi}^a_L = \phi^a + U_L{}^a(\varphi, \phi)\,, \quad \tilde{\omega}^\alpha_L = V_L{}^\alpha_\beta(\varphi, \phi)\,\omega^\beta$$

independent of other sources



I think you should be a little
more specific, here in Step 2

Slavnov-Taylor and Ward identities

Derivation of ST identities: $\boldsymbol{Q}_{\text{ext}}\Sigma_{\text{ext}} = 0 \;\Rightarrow\; \underbrace{\int d\Phi\, \boldsymbol{Q}_{\text{ext}}}_{\displaystyle \| \atop \displaystyle -J\frac{\delta}{\delta\gamma} + \bar{\xi}\frac{\delta}{\delta\zeta} + \xi\frac{\delta}{\delta y} + \Omega\frac{\delta}{\delta\phi} + \boldsymbol{s}} e^{-\Sigma_{\text{ext}}/\hbar} = 0$

$$\left(-J\frac{\delta}{\delta\gamma} + \bar{\xi}\frac{\delta}{\delta\zeta} + \xi\frac{\delta}{\delta y} + \Omega\frac{\delta}{\delta\phi}\right) e^{-W/\hbar}$$
$$= -\int d\Phi\, \boldsymbol{s}\, e^{-\Sigma_{\text{ext}}/\hbar} = \int d\Phi\, \underbrace{\left(\frac{\delta}{\delta\Phi^I}\boldsymbol{s}\Phi^I(\Phi)\right)}_{=0} e^{-\Sigma_{\text{ext}}/\hbar}$$

Assumption of anomaly-free regularization

Derivation of Ward identities:

$$0 = \int d\Phi\, \delta_\varepsilon e^{-\Sigma_{\text{ext}}/\hbar} = \int d\Phi\, \Bigg(\underbrace{\delta_\varepsilon\Phi\frac{\delta}{\delta\Phi}}_{=0} + \delta_\varepsilon\phi\frac{\delta}{\delta\phi} + \delta_\varepsilon\mathcal{J}\frac{\delta}{\delta\mathcal{J}}\Bigg) e^{-\Sigma_{\text{ext}}/\hbar}$$
$$= \left(\delta_\varepsilon\phi\frac{\delta}{\delta\phi} + \delta_\varepsilon\mathcal{J}\frac{\delta}{\delta\mathcal{J}}\right) e^{-W/\hbar}$$

all sources
$\mathcal{J} = J,\, \bar{\xi},\, \xi,\, y,\, \gamma,\, \zeta,\, \Omega$

BRST $\rightarrow$

$$\left[-J\frac{\delta}{\delta\gamma}+\bar{\xi}\frac{\delta}{\delta\zeta}+\xi\frac{\delta}{\delta y}+\Omega\frac{\delta}{\delta\phi}\right]W=0$$

BGT $\rightarrow$

$$\left[\delta_\varepsilon\phi\frac{\delta}{\delta\phi}+\delta_\varepsilon\mathcal{J}\frac{\delta}{\delta\mathcal{J}}\right]W=0$$

all sources

$$\mathcal{J}=J,\ \bar{\xi},\ \xi,\ y,\ \gamma,\ \zeta,\ \Omega$$

b_α multiplier equation $\rightarrow$

$$\left[\chi^\alpha_a\frac{\delta}{\delta J_a}-O^{\alpha\beta}\frac{\delta}{\delta y^\beta}-\frac{\Omega^a}{2}\frac{\delta O^{\alpha\beta}}{\delta\phi^a}\frac{\delta}{\delta\xi^\beta}+y^\alpha\right]W=0$$

Transition to effective action $\quad W\rightarrow\Gamma\Big[\langle\varphi\rangle,\langle\omega\rangle,\langle\bar{\omega}\rangle,\langle b\rangle,\phi,\gamma,\zeta,\Omega\Big]$

$$\Gamma\Big[\langle\varphi\rangle,\langle\omega\rangle,\langle\bar{\omega}\rangle,\langle b\rangle,\phi,\gamma,\zeta,\Omega\Big]=W-J_a(\langle\varphi^a\rangle-\phi^a)-\bar{\xi}_\alpha\langle\omega^\alpha\rangle-\xi^\alpha\langle\bar{\omega}_\alpha\rangle-y^\alpha\langle b_\alpha\rangle$$

reduced effective action

$$\Gamma\rightarrow\hat{\Gamma}=\Gamma-\left(b_\alpha+\Omega^a\bar{\omega}_\alpha\frac{\delta}{\delta\phi^a}\right)\left(\chi^\alpha_a(\varphi-\phi)^a-\frac{1}{2}O^{\alpha\beta}b_\beta\right)$$

subtracting gauge fixing term

$$\Downarrow$$

$$\widehat{\Gamma} = \widehat{\Gamma}[\varphi, \omega, \phi, \widehat{\gamma}, \zeta, \Omega], \quad \widehat{\gamma}_a = \gamma_a - \bar{\omega}^\alpha \chi^a_\alpha, \quad \frac{\delta \widehat{\Gamma}}{\delta b} = 0$$

Slavnov-Taylor and Ward identities for $\widehat{\Gamma}$

$$\frac{\delta \widehat{\Gamma}}{\delta \widehat{\gamma}_a} \frac{\delta \widehat{\Gamma}}{\delta \varphi^a} + \frac{\delta \widehat{\Gamma}}{\delta \zeta_\alpha} \frac{\delta \widehat{\Gamma}}{\delta \omega^\alpha} + \Omega^a \frac{\delta \widehat{\Gamma}}{\delta \phi^a} = 0 \quad \Leftrightarrow \quad (\widehat{\Gamma}, \widehat{\Gamma}) + \Omega^a \frac{\delta \widehat{\Gamma}}{\delta \phi^a} = 0$$

antibracket

$$\vec{\boldsymbol{W}} \widehat{\Gamma} \equiv R^a{}_\alpha(\varphi) \frac{\delta \widehat{\Gamma}}{\delta \varphi^a} - C^\gamma{}_{\beta\alpha} \omega^\beta \frac{\delta \widehat{\Gamma}}{\delta \omega^\gamma} + R^a{}_\alpha(\phi) \frac{\delta \widehat{\Gamma}}{\delta \phi^a} - \widehat{\gamma}_b R^b{}_{a\alpha} \frac{\delta \widehat{\Gamma}}{\delta \widehat{\gamma}_a}$$

$$+ \zeta_\beta C^\beta{}_{\gamma\alpha} \frac{\delta \widehat{\Gamma}}{\delta \zeta_\gamma} + R^a{}_{b\alpha} \Omega^b \frac{\delta \widehat{\Gamma}}{\delta \Omega^a} = 0$$

Structure of divergences: tale of two (many) BRST operators

Proceed perturbatively in $\hbar$:

$$\hat{\Gamma} = \hat{\Sigma}_0 + O(\hbar)$$

$$\hat{\Sigma}_0 = S[\varphi] + \hat{\gamma}_a R^a{}_\alpha(\varphi)\,\omega^\alpha + \frac{1}{2}\zeta_\alpha C^\alpha{}_{\beta\gamma}\omega^\beta\omega^\gamma$$

$\Gamma_L,\quad \Gamma_L^{\rm div} = \mathcal{O}(\hbar^{L+1})$ *-- action renormalized in the L-th loop order*

By induction $\Gamma_{L-1} \to \Gamma_L$ $\qquad \Gamma_L^{\rm div} \equiv \Gamma_{L,\infty} = \hat{\Gamma}_\infty$

$$(\hat{\Gamma},\hat{\Gamma}) + \Omega^a \frac{\delta\hat{\Gamma}}{\delta\phi^a} = 0 \quad \Rightarrow \quad \boxed{\boldsymbol{Q}_+\hat{\Gamma}_\infty = 0}$$

$$\boldsymbol{Q}_+ X[\varphi,\omega,\phi,\hat{\gamma},\zeta,\Omega] \equiv (\hat{\Sigma}_0, X) + \Omega^a\frac{\delta X}{\delta\phi^a} = 0. \quad \boldsymbol{Q}_+ \neq \boldsymbol{Q}$$

minimal sector

$$\boldsymbol{Q}_+^2 = 0 \qquad \textit{nilpotent}$$

trivial cohomology of $\Omega\frac{\delta}{\delta\phi}$

Batalin and Vilkovisky (1985)

decoupling of invariant counterterm

$$\hat{\Gamma}_\infty = \boldsymbol{S}[\varphi] + \Lambda[\varphi,\omega,\hat{\gamma},\zeta] + \boldsymbol{Q}_+\Upsilon$$

$$\frac{\delta \boldsymbol{S}}{\delta\varphi^a}R^a{}_\alpha(\varphi) = 0$$

single field counterterm

ghost term

$$\Lambda\Big|_{\omega=0} = 0$$

BRST exact term (not yet the original one!)

Applies to nonrenormalizable EFT within gradient expansion: truncate in number of derivatives + locality + Grassman statistics of Ω

ST+W identities: $(q_0+q_1)\,\Lambda = 0,\ \ (q_0)^2 = (q_1)^2 = q_0q_1 + q_1q_0 = 0$

$$\underbrace{q_0 = \frac{\delta S}{\delta\varphi^a}\frac{\delta}{\delta\hat{\gamma}_a} - \hat{\gamma}_a R^a{}_\alpha(\varphi)\frac{\delta}{\delta\zeta_\alpha},}_{} \qquad q_1 = -\frac{1}{2}C^\gamma{}_{\alpha\beta}\omega^\alpha\omega^\beta\frac{\delta}{\delta\omega^\gamma}$$

Kozhul-Tate differential has a trivial cohomology under the assumption of local completeness and irreducibility of gauge generators for $\Lambda\Big|_{\omega=\hat{\gamma}=\zeta=0} = 0$

$$\Lambda = \boldsymbol{Q}_+\Xi$$

Henneaux (1991)
Vandoren, Van Proeyen (1994)

Thus:

BRST exact term (not yet the original BRST charge!)

$$\hat{\Gamma}_\infty = \boldsymbol{\Gamma}[\varphi, \omega, \hat{\gamma}, \zeta] + \boldsymbol{Q}_+ \Upsilon$$

$$\boldsymbol{\Gamma} = \boldsymbol{S}[\,\varphi\,] + \boldsymbol{Q}_+ \Xi$$

$$\hat{\Gamma}_\infty = \Gamma_\infty$$

full action vs reduced

$$\Gamma_{L,\infty} = \boldsymbol{S}_L[\varphi] + \boldsymbol{Q}_+ \boldsymbol{\Upsilon}_L, \qquad \boldsymbol{\Upsilon}_L = \Upsilon_L + \Xi_L$$

at any loop order *L*

L-th order subtraction and $Q_+ \to Q$ transition via field redefinition

$$\Sigma_L[\Phi, \phi, \gamma, \zeta, \Omega] = \Sigma_{L-1} - \hbar^L \Gamma_{L,\infty} + \mathcal{O}(\hbar^{L+1})$$

gauge fermion renormalization

$$\Psi_L = \Psi_{L-1} - \hbar^L \boldsymbol{\Upsilon}_L$$

$$\Psi_L = \Psi_0 - \sum_{l=1}^{L} \hbar^l \boldsymbol{\Upsilon}_L$$

field redefinition $\Phi = \varphi, \omega, \ldots \to \Phi' = \varphi', \omega', \ldots$:

$$\Sigma_L[\Phi, \phi, \gamma, \zeta, \Omega] = \Big[\Sigma_{L-1} - \hbar^L \boldsymbol{S}_L - \hbar^L \boldsymbol{Q}\, \boldsymbol{\Upsilon}_L \Big]_{\Phi \to \Phi'}$$

$$\tilde{\varphi}_{L-1}(\Phi) - \phi = -\frac{\delta \Psi_L(\Phi')}{\delta \gamma}\,, \quad \tilde{\omega}_{L-1}(\Phi) = \frac{\delta \Psi_L(\Phi')}{\delta \zeta}$$

Gauge fermion is a generating function of the field redefinition

Example: 2D O(N) gauge model

$$S = \frac{1}{2g^2}\int \mathrm{d}^2x\,\partial_\mu n_i \partial^\mu n^i,\ \ i = 1,...N,\ \ n^2 = 1$$

$$n^i = \frac{\varphi^i}{\sqrt{\varphi^2}} \quad \Rightarrow \quad S[\varphi] = \frac{1}{2g^2}\int \mathrm{d}^2x\, G_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j,$$

$$G_{ij} = \frac{1}{\varphi^2}\left[\delta_{ij} - \frac{\varphi_i\varphi_j}{\varphi^2}\right]$$

Abelian gauge invariance

$$\delta_\varepsilon\varphi^i(x) = \varphi^i(x)\,\varepsilon(x)$$

$$R^a{}_{b\alpha} \mapsto \delta^i_j\,\delta(x-x_1)\,\delta(x-x_2)$$

Gauge fixing

$$\chi^\alpha(\varphi,\phi) \mapsto \chi = \Box\left(\frac{\phi_i(x)\left(\varphi^i(x) - \phi^i(x)\right)}{\phi^2(x)}\right)$$

$$O^{\alpha\beta}(\phi) \mapsto O(x,x') = -\Box\,\delta(x-x'), \quad O^{-1}_{\alpha\beta}(\phi) \mapsto O^{-1}(x,x') = -\frac{1}{\Box}\delta(x-x')$$

$$\Psi_0 = \int \mathrm{d}^2x\left(-\hat{\gamma}_i\,(\varphi^i - \phi^i) + \zeta\omega\right) + ...,\ \ \hat{\gamma}_i = \gamma_i - \frac{\phi_i}{\phi^2}\Box\bar{\omega}$$

$$\Sigma_0[\varphi,\omega,\bar{\omega},\phi,\gamma,\Omega] = \int \mathrm{d}^2x \left\{ \frac{1}{2} G_{ij}\,\partial_\mu\varphi^i\partial^\mu\varphi^j - \frac{1}{2}\frac{\varphi\cdot\phi}{\phi^2}\Box\left(\frac{\varphi\cdot\phi}{\phi^2}\right) - \frac{\varphi\cdot\phi}{\phi^2}(\Box\bar{\omega})\,\omega \right.$$
$$\left. +(\gamma\cdot\varphi)\omega + \left(\frac{\Omega\cdot\varphi}{\phi^2} - 2\frac{(\varphi\cdot\phi)\,(\Omega\cdot\phi)}{(\phi^2)^2}\right)\Box\bar{\omega} + \Omega\cdot\gamma \right\}$$

One-loop divergences

$$\hbar\Gamma_{1,\infty} = -\frac{\hbar}{2\pi(2-d)}\int \mathrm{d}^2x \left\{ \frac{N-2}{2} G_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j + \frac{(\phi^2)^2}{(\varphi\cdot\phi)^2}(\varphi\cdot\hat{\gamma})\,\omega \right.$$
$$+\left(\frac{\delta_{ij}}{\varphi^2} - 2\frac{\phi_i\varphi_j}{(\varphi\cdot\phi)\varphi^2} + \frac{\phi_i\phi_j}{(\varphi\cdot\phi)^2}\right)\partial_\mu\varphi^i\partial^\mu\varphi^j - \left(\frac{\delta_{ij}}{(\varphi\cdot\phi)} - \frac{\phi_i\varphi_j}{(\varphi\cdot\phi)^2}\right)\partial_\mu\varphi^i\partial^\mu\phi^j$$
$$\left. \underbrace{-\left[\frac{\varphi^2}{(\varphi\cdot\phi)}\delta^i_k - 2\,\frac{\phi^2}{(\varphi\cdot\phi)^2}\varphi^i\phi_k - \frac{\varphi^2}{(\varphi\cdot\phi)^2}\left(\phi^i\varphi_k + \varphi^i\phi_k\right) + \frac{\phi^2(\varphi^2+\phi^2)}{(\varphi\cdot\phi)^3}\varphi^i\varphi_k\right]\Omega^k\hat{\gamma}_i}_{\sim\ \Omega\,\delta\boldsymbol{\Upsilon}_1/\delta\phi} \right\}.$$

$$\boldsymbol{\varUpsilon}_1 = \int d^2x\, \widehat{\gamma}_i \boldsymbol{u}_1^i(\varphi,\phi)$$

$$\boldsymbol{u}_1^i(\varphi,\phi) = -\frac{1}{4\pi(2-d)}\left[\frac{\phi^2(\varphi^2+\phi^2)}{(\varphi\cdot\phi)^2}\varphi^i - \frac{2\varphi^2}{(\varphi\cdot\phi)}\phi^i\right]$$

Field redefinition

$$\varphi^i \mapsto \tilde{\varphi}_1^i = \varphi^i + \hbar \boldsymbol{u}_1^i(\varphi,\phi)$$

essentially nonlinear

$$\varSigma_1 \equiv \left[\,\varSigma_0 - \hbar \varGamma_{1,\infty}\right]_{\varphi\to\varphi+\hbar\boldsymbol{u}_1}$$

$$= S[\varphi] - \hbar\,\boldsymbol{S}_1[\varphi] + \boldsymbol{Q}\left(\varPsi_0 - \hbar\boldsymbol{\varUpsilon}_1\right)$$

Physical one-loop counterterm

$$\boldsymbol{S}_1[\varphi] = -\frac{N-2}{4\pi(2-d)}\int \mathrm{d}^2x\, G_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j$$

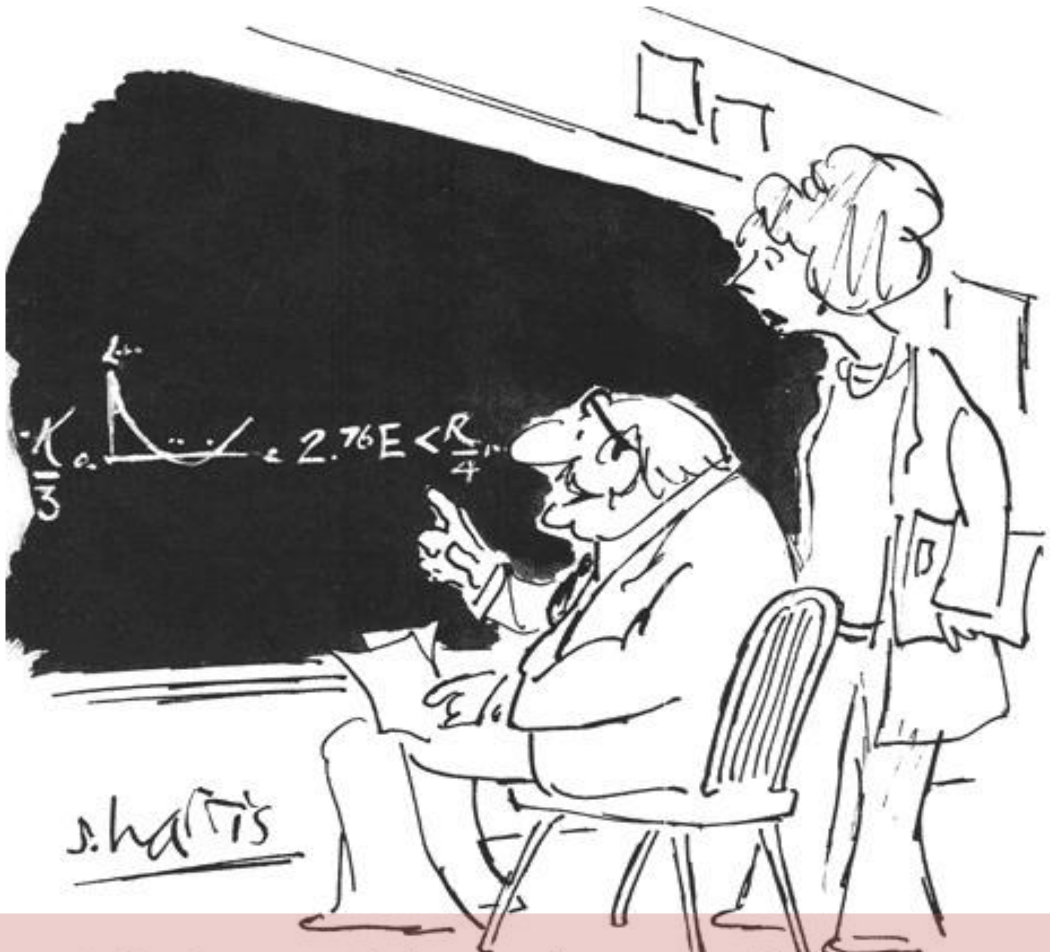
Conclusions and Outlook

- Background field method is not only a convenient calculational tool, but is also efficient for general analysis of the structure of renormalization

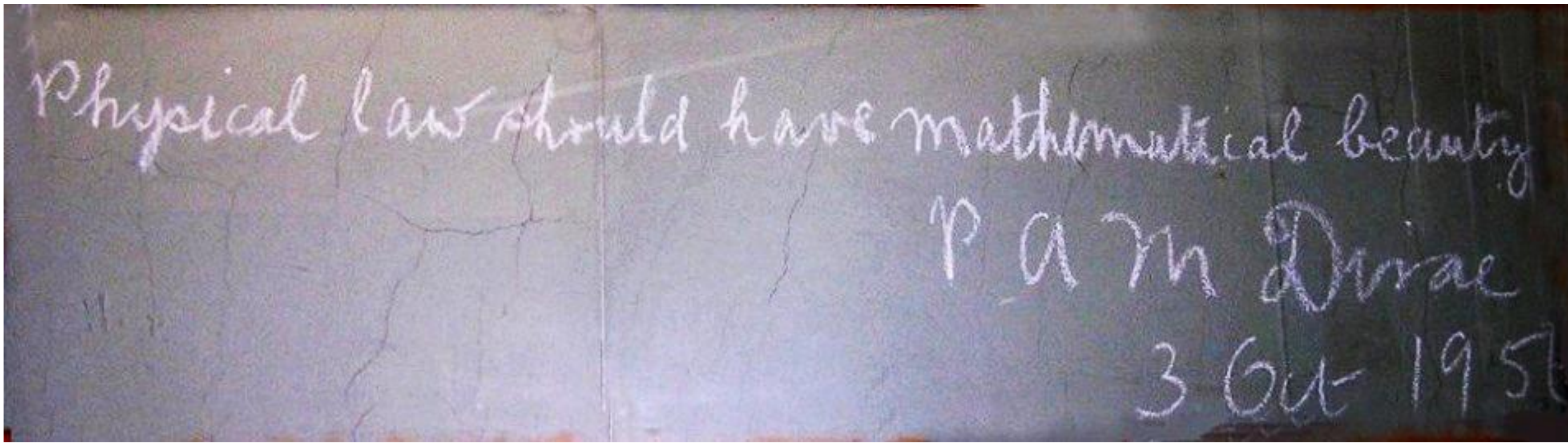
- BRST structure (gauge invariance) is preserved by renormalization for non-anomalous theories whose gauge algebra:
  i) has linear generators
  ii) closes off-shell *can be relaxed (?)*
  iii) is locally complete
  iv) is irreducible *can be relaxed*

- Generalizations: open algebras, supersymmetry, composite operators, anomalies

The power and beauty of the nilpotent BRST operator



"The beauty of this is that it is only of theoretical importance, and there is no way it can be of any practical use whatsoever."


Physical law should have mathematical beauty
P A M Dirac
3 Oct 1956