BLTP, JINR

Fractional Hopfions

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Outline

- Baby Skyrme model and Faddeev-Skyrme model
- Rational maps and Hopfions
- Symmetry breaking potentials
- Fractionally charged Hopfions
- Gauged Hopfions
- Conclusion and outlook

Skyrme family

- ***(2+1)-dim***: **Baby Skyrme model**

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{4}\left(\partial_\mu \phi \times \partial_\nu \phi\right)^2 - V(\phi)}$$

$$\phi : S^2 \to S^2; \quad \phi_\infty = (0,0,1)$$

Standard choice: $V(\phi) = \mu^2(1-\phi_3)$

$$Q \in \mathbb{Z} = \pi_2(S^2)$$

$$Q = \frac{1}{4\pi}\int_{\mathbb{R}^2} \phi \cdot (\partial_1 \phi \times \partial_2 \phi) d^2x$$

- ***(3+1)-dim***: **Faddeev-Skyrme model**

$$\phi : S^3 \to S^2; \quad \phi_\infty = (0,0,1)$$

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{4}\left(\partial_\mu \phi \times \partial_\nu \phi\right)^2 - V(\phi)}$$

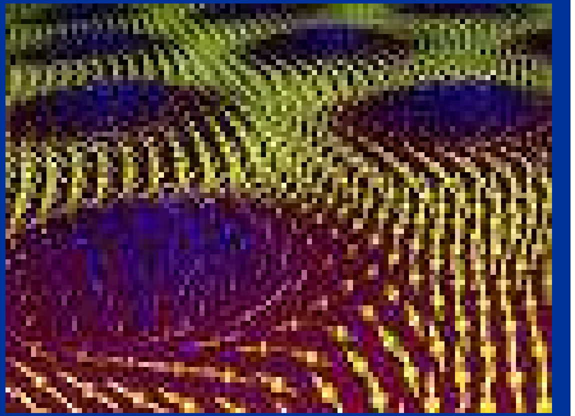
$$Q \in \mathbb{Z} = \pi_3(S^2)$$

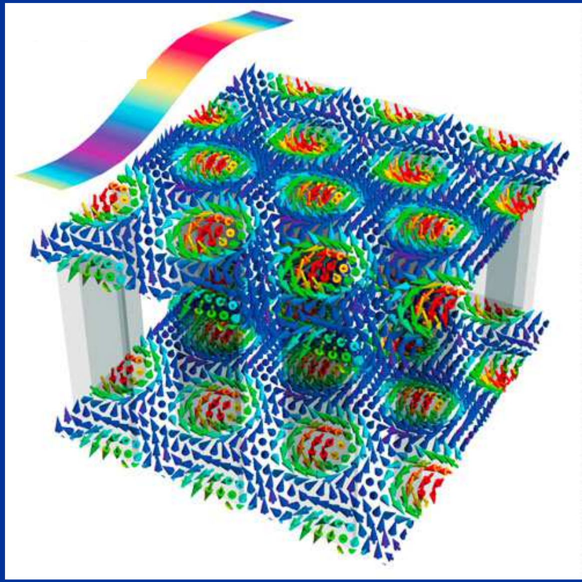
$$\phi^1 + i\phi^2 \to (\phi^1 + i\phi^2)e^{i\alpha}$$

Residual SO(2) symmetry

Baby Skyrme model: Applications

- A Heisenberg-type model of interacting spins
- A model of the topological quantum Hall effect
- Chiral magnetic structures
- A model of ferromagnetic planar structures
- Applications in future development of data storage technologies
- Models of condensed matter systems with intrinsic and induced chirality





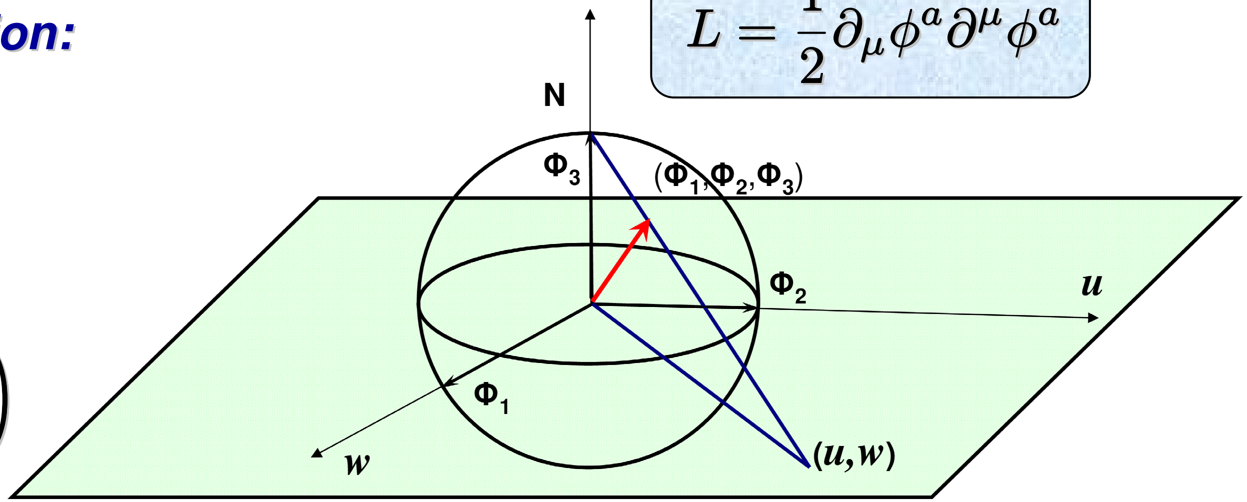
Rößler et al.
Nature 442 (2006) 797

O(3) sigma-model vs $\mathbb{CP}^1$ model

- ***Stereographic projection:***

$$L = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a$$

Coordinates on the projective space $\mathbb{CP}^1$



$$(u,w) = \left(\frac{\phi_1}{1-\phi_3}, \frac{\phi_2}{1-\phi_3}\right)$$

$$Z = u + iw = \frac{\phi_1 + i\phi_2}{1-\phi^3}$$

$z = x + iy$

Domain space:

Target space:

Inverse transformation onto $\mathbb{S}^2$

$$(\phi_1,\phi_2,\phi_3) = \left(\frac{2u}{1+u^2+w^2}, \frac{2w}{1+u^2+w^2}, \frac{1-u^2-w^2}{1+u^2+w^2}\right)$$

$$= \left(\frac{Z+\bar{Z}}{1+Z\bar{Z}}, i\frac{\bar{Z}-Z}{1+Z\bar{Z}}, \frac{1-Z\bar{Z}}{1+Z\bar{Z}}\right)$$

$\mathbb{CP}^1$ model

- ***Lagrangian:*** $L = \dfrac{\partial_\mu Z \partial^\mu \bar{Z}}{(1+Z\bar{Z})^2}$
- ***Metric:*** $dS^2 = (d\phi_1)^2 + (d\phi_2)^2 + (d\phi_3)^2 = 4\dfrac{dZd\bar{Z}}{(1+Z\bar{Z})^2}$
- ***Kinetic energy:*** $T = \dfrac{|\partial_t Z|^2}{(1+Z\bar{Z})^2}$
- ***Potential energy:*** $V = \dfrac{|\partial_i Z|^2}{(1+Z\bar{Z})^2}$

Holomorphic derivatives:

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right); \qquad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right); \qquad \boxed{ds^2 = dzd\bar{z}}$$

$$\boxed{V = \frac{|Z_z|^2 + |Z_{\bar{z}}|^2}{(1+|Z|^2)^2}}$$

Energy

$$\boxed{Z_{z\bar{z}} = 2\bar{Z}\frac{Z_z Z_{\bar{z}}}{(1+|Z|^2)}}$$

Field equations

$$\boxed{Q = \frac{|Z_z|^2 - |Z_{\bar{z}}|^2}{(1+|Z|^2)^2}}$$

Topological charge density

$\mathbb{CP}^1$ model: Solitons

$$E = \int \frac{|Z_z|^2 + |Z_{\bar z}|^2}{(1+|Z|^2)^2} dz d\bar z$$

The energy is minimal if $Z_{\bar z} = 0$

Cauchy-Riemann conditions for Z

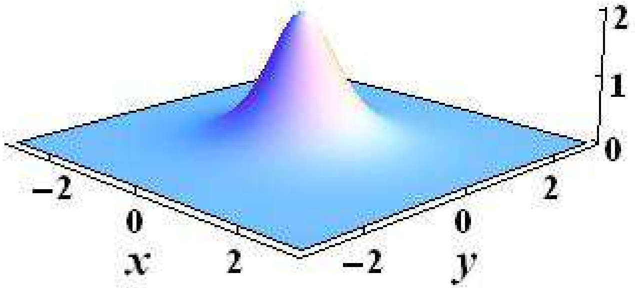
Simplest holomorphic solution:

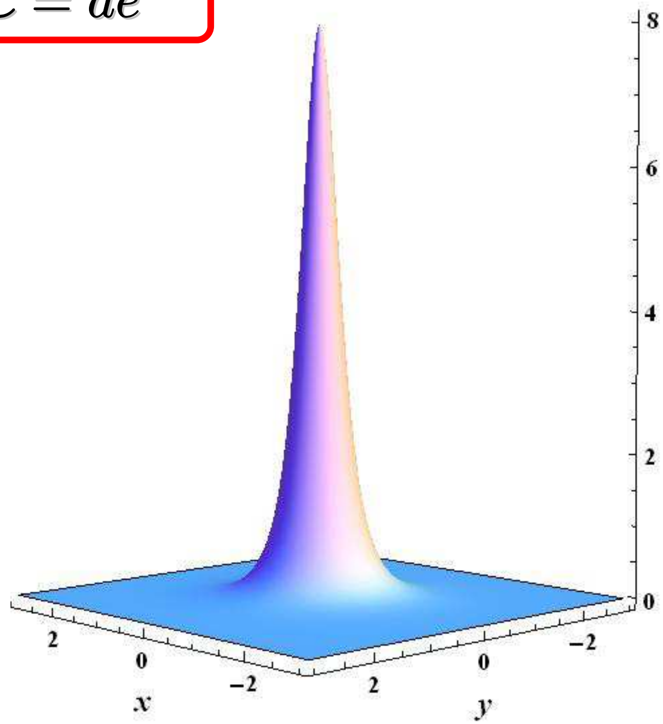
$$\boxed{Z = \lambda z; \quad \lambda \in \mathbb{C} = a e^{i\delta}}$$

Rational map holomorphic solution of degree 1:

$$Z = \frac{P(z)}{Q(z)} = \frac{\lambda(z-a)}{z-b}$$

Q=1:

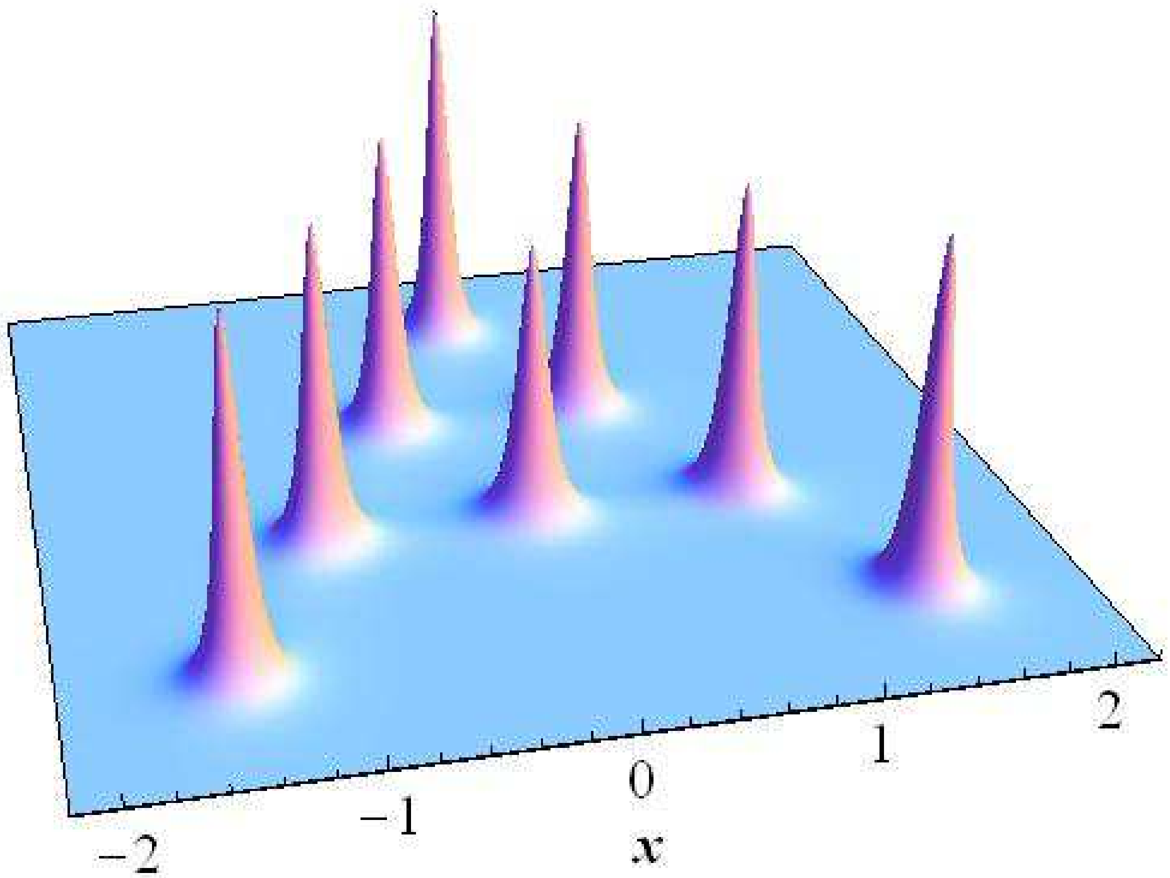




$\mathbb{CP}^1$ model: Solitons

Rational map holomorphic solution of degree 8: $Z = \dfrac{P(z)}{Q(z)}$

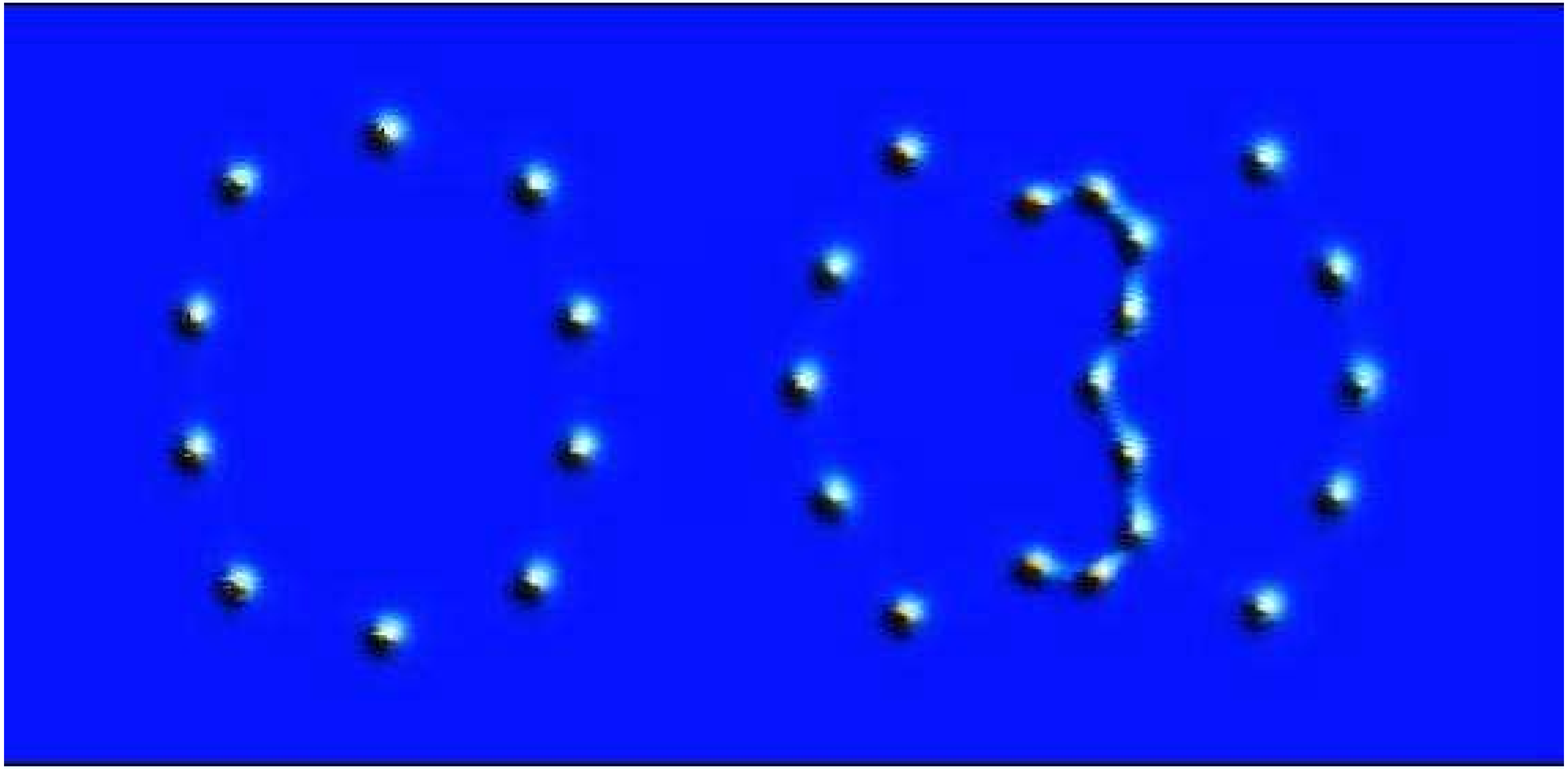
$$Z(z) = \frac{4}{\frac{1}{z} + \frac{1}{z+\frac{1}{2}-i} + \frac{1}{z-\frac{1}{2}-i} + \frac{1}{z-1} + \frac{1}{z-1} + \frac{1}{z+\frac{3}{2}+i} + \frac{1}{z-\frac{3}{2}+i} + \frac{1}{z-2i}}$$



$\mathbb{CP}^1$ model: Solitons

Rational map holomorphic solution of degree 29:

$$Z = \frac{P(z)}{Q(z)}$$

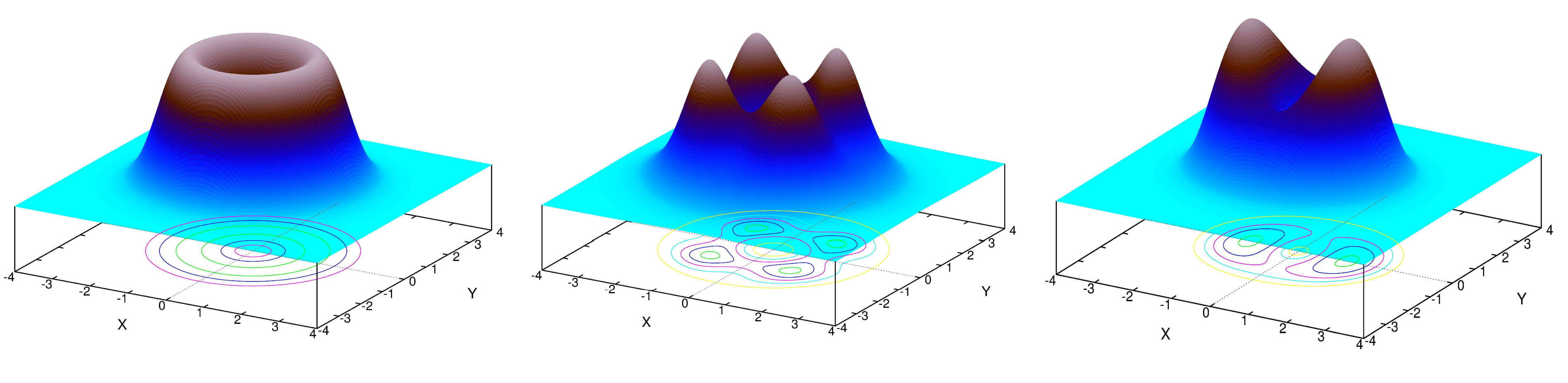


Baby Skyrme model

Potential of the baby Skyrme model: potential term U(ϕ) may be chosen almost arbitrarily, however must vanish at infinity for a given vacuum field value in order to ensure existance of the finite energy solutions: $\phi^a_{(0)} = (0, 0, 1)$

Several potential terms have been studied in great detail:

- "Old" model, with $U(\phi) = m^2(1 - \phi_3)$
- Holomorphic model, with $U(\phi) = m^2(1 - \phi_3)^4$
- "Double vacuum" model, with $U(\phi) = m^2(1 - \phi_3^2)$



Karliner, Hen (2007) $U(\phi) = m^\alpha(1 - \phi_3^\beta)$

Baby Skyrme model: solitons

Rotationally invariant ansatz:

$$\phi^1 = \sin f(r) \cos\varphi; \quad \phi^2 = \sin f(r)\sin\varphi; \quad \phi^3 = \cos f(r)$$

Q=1

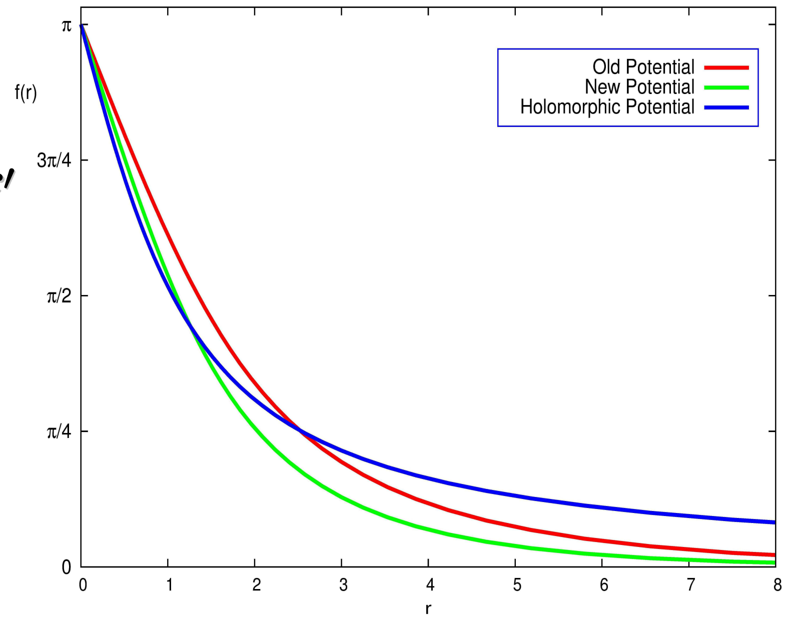
$$U(\phi) = \mu^2(1-\phi_3)$$

- **Energy:** $$E = 2\pi \int\limits_0^\infty rdr \left(\frac{1}{2}f'^2 + \frac{\sin^2 f}{2r^2}(f'^2+1) + \mu^2(1-\cos f)\right)$$
- **Topological charge:** $$Q = \frac{1}{2}\int\limits_0^\infty rdr\left(\frac{f'\sin f}{r}\right) = \frac{1}{2}[\cos f(0) - \cos f(\infty)]$$
- **Field equation:**

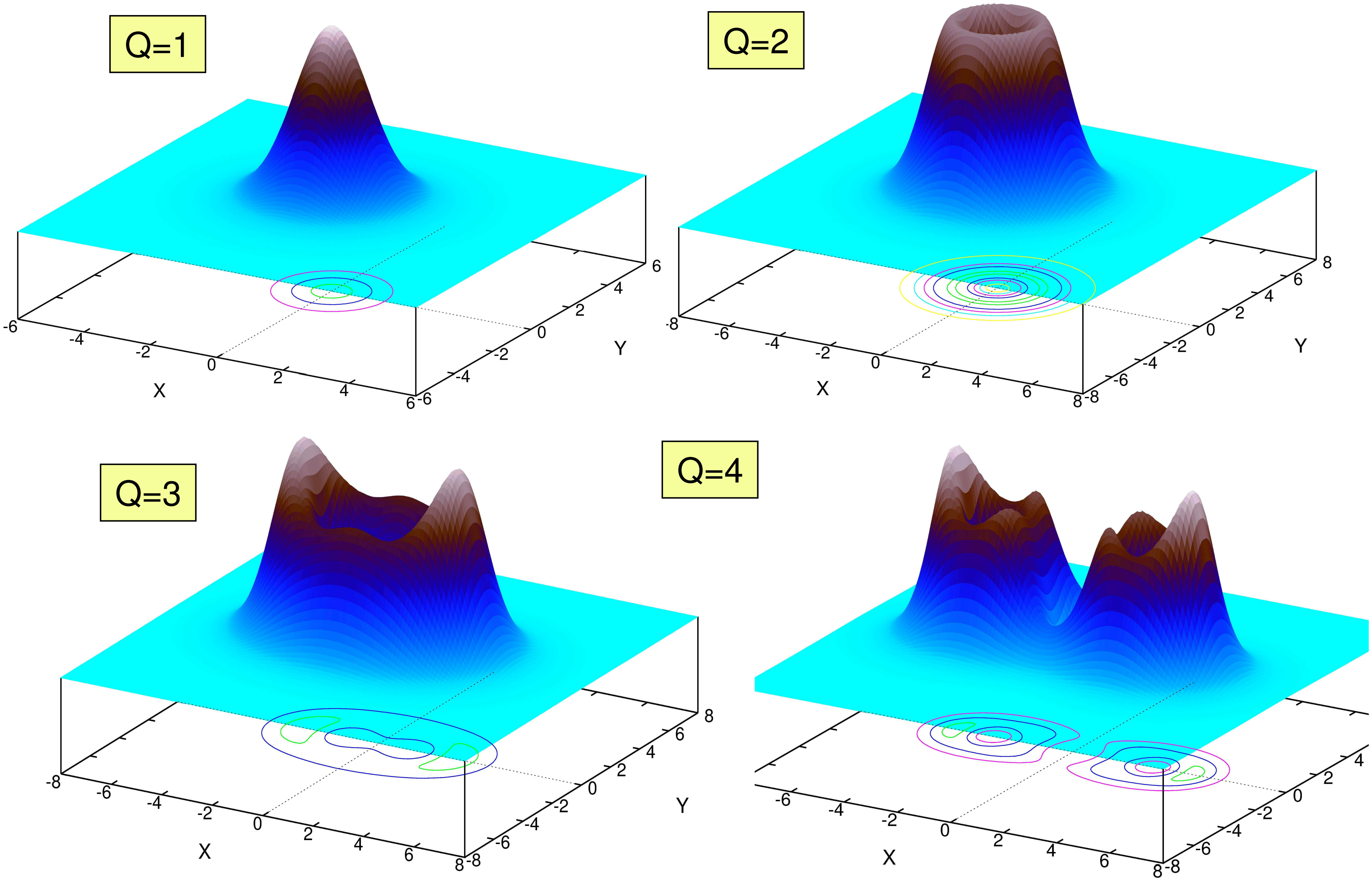
$$\left(r + \frac{\sin^2 f}{r}\right) f'' + \left(1 - \frac{\sin^2 f}{r^2} + \frac{f'\sin f\cos f}{r}\right) f' - \frac{\sin f\cos f}{r} - r\mu^2 \sin f = 0$$

- **Linearized field equation:**

$$f'' + \frac{1}{r}f' - \left(\mu^2 + \frac{1}{r^2}\right) f = 0$$

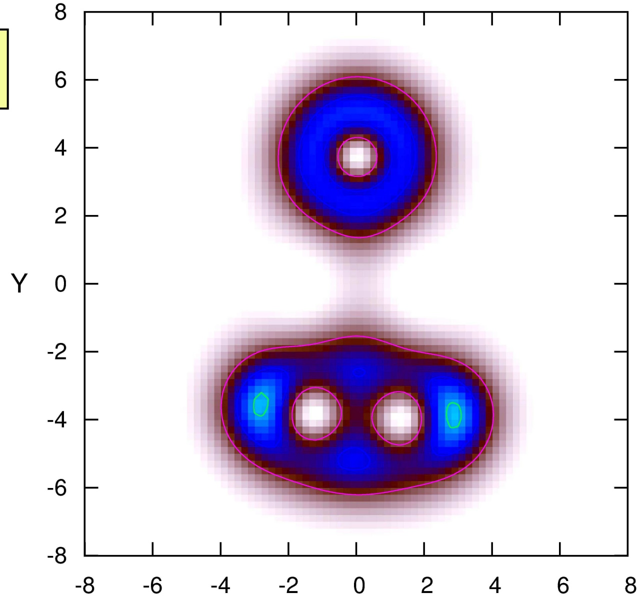


Baby Skyrme model: solitons

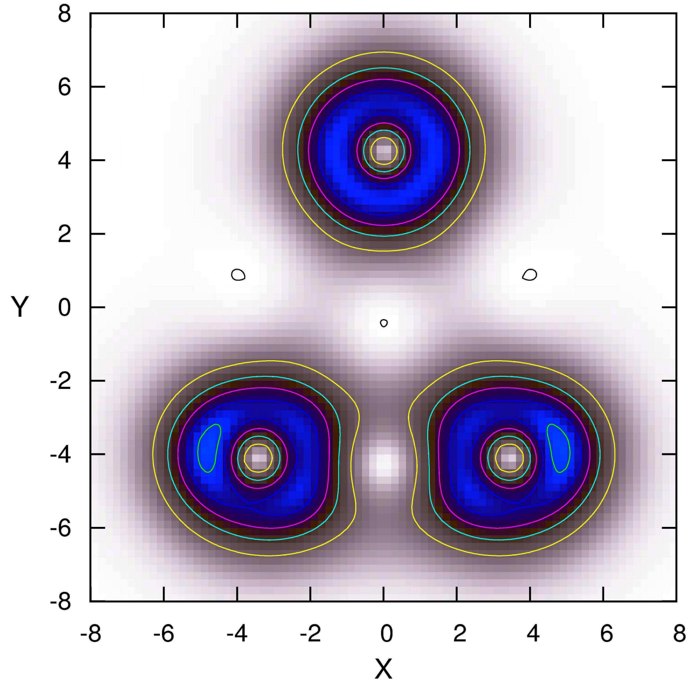


Baby Skyrme model: solitons

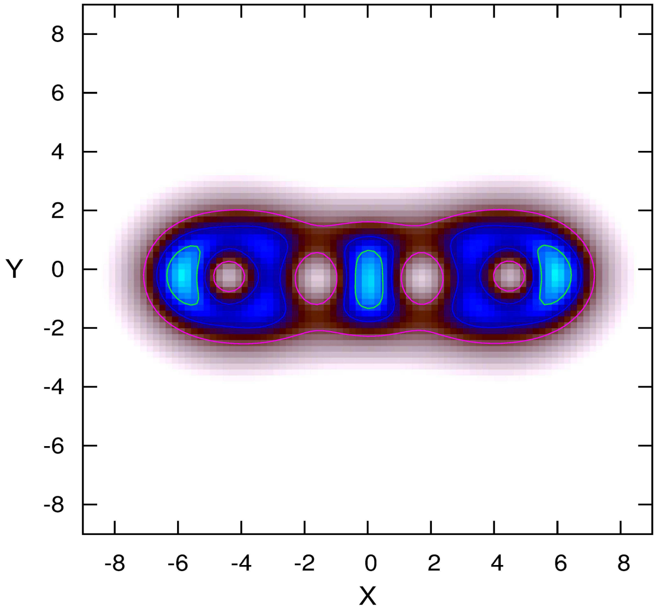
Q=5



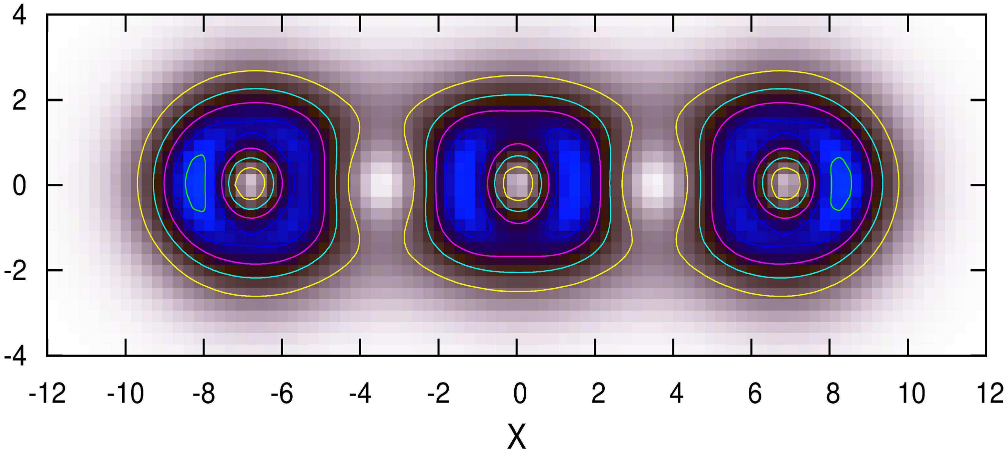
Q=6



Q=5

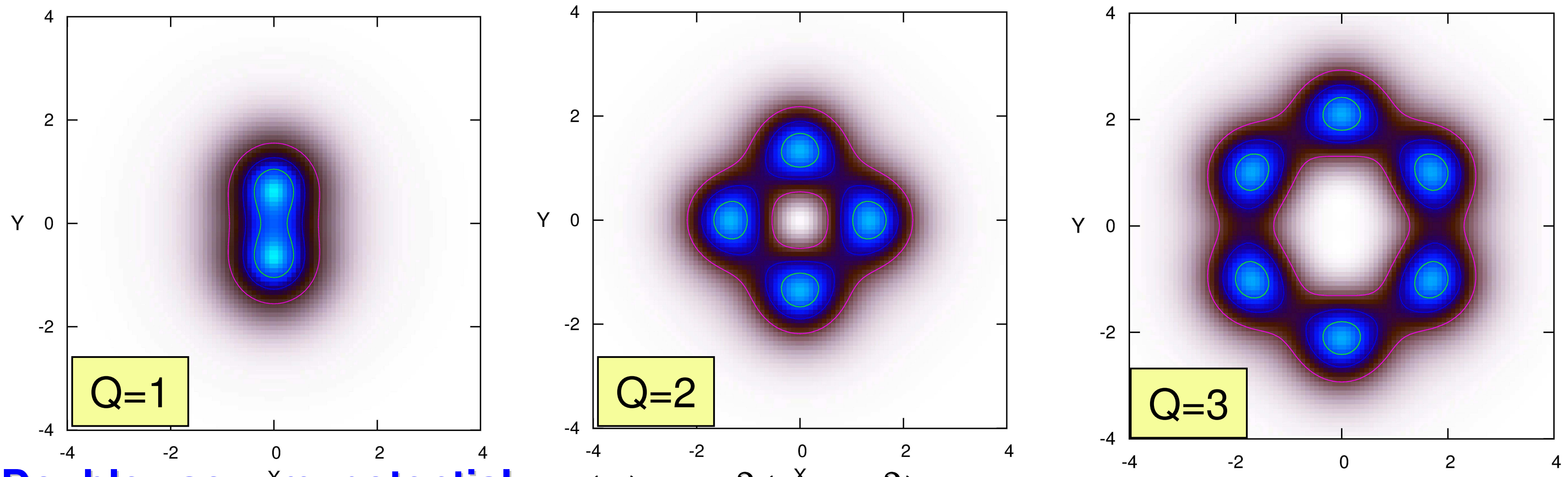


Q=6

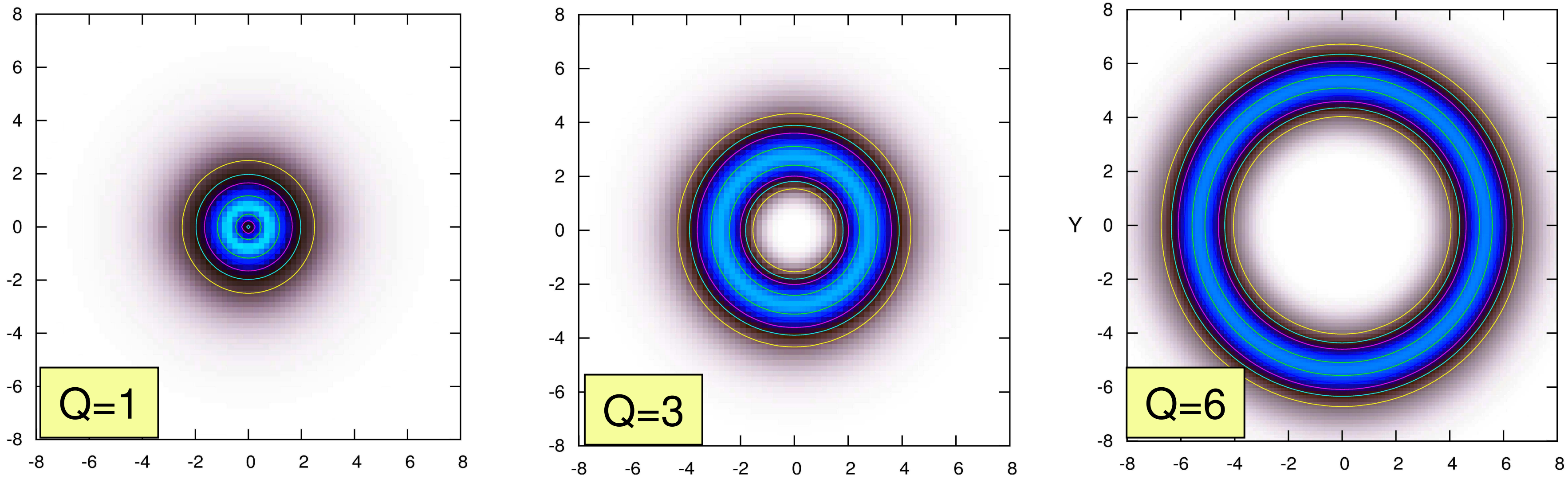


Baby Skyrme model: solitons

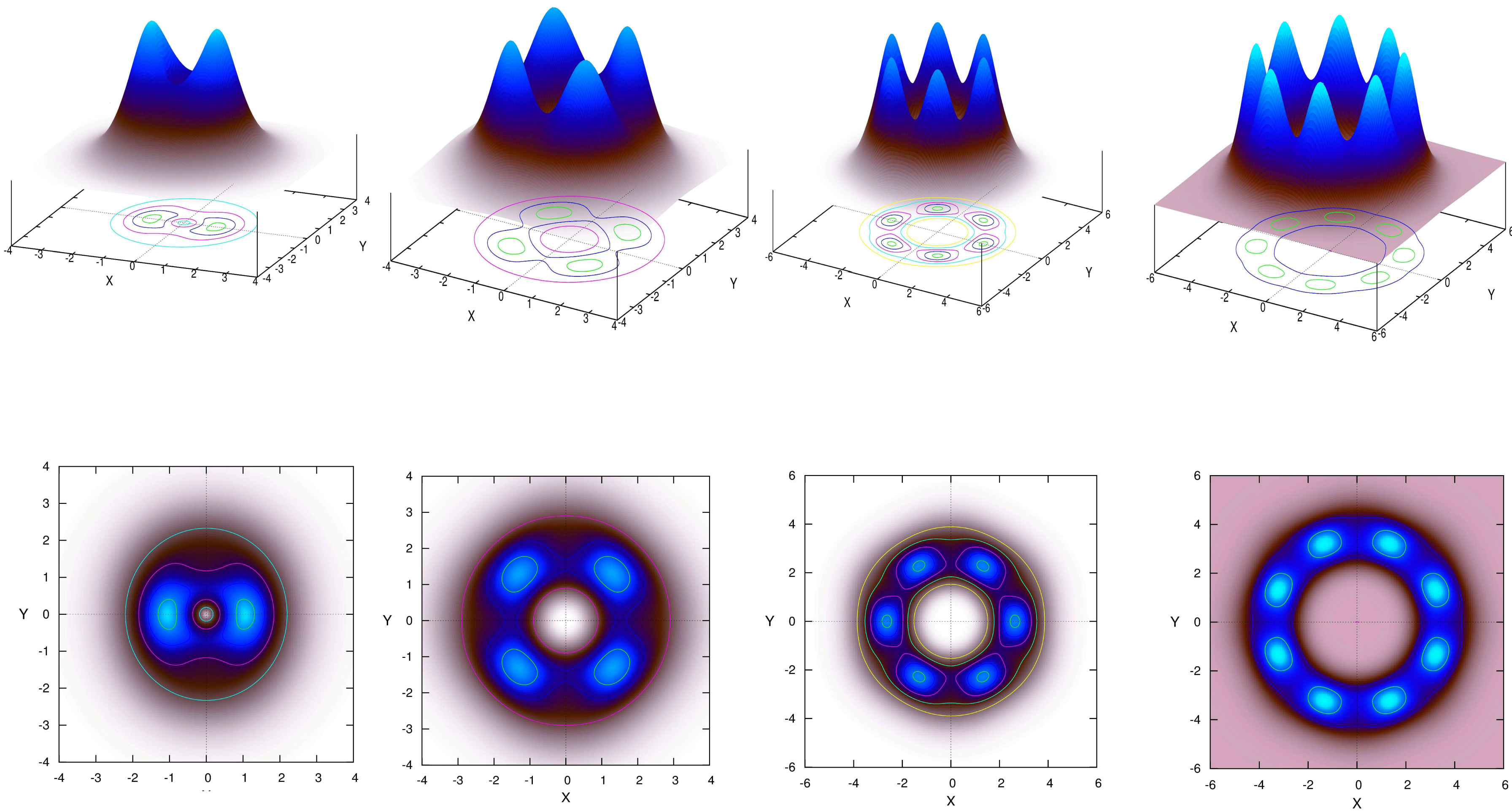
- **Easy plane potential** $U(\phi) = \mu^2 \phi_1^2$



- **Double vacuum potential** $U(\phi) = \mu^2 (1 - \phi_3^2)$



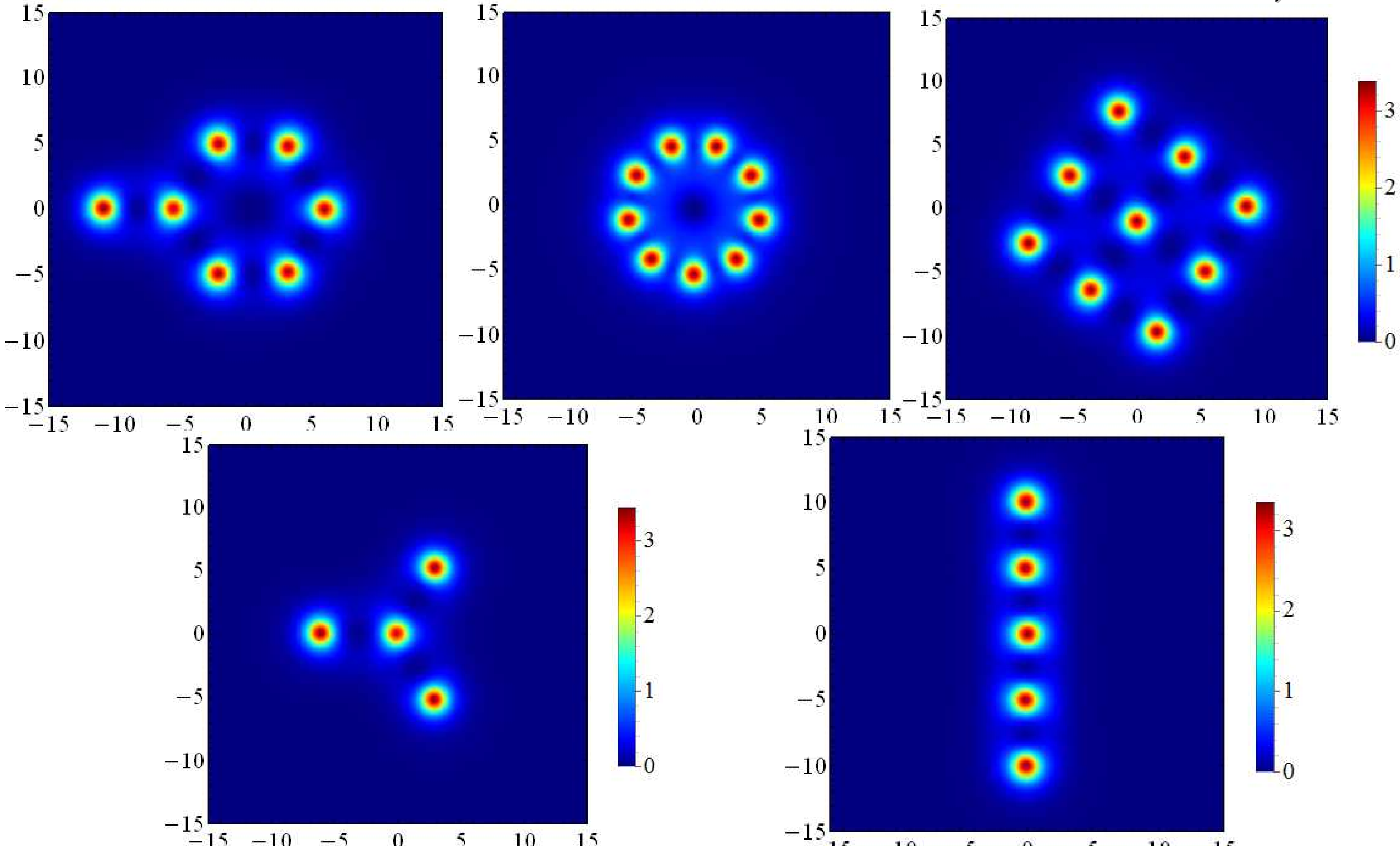
Symmetry breaking Ward potential: $U(\phi) = m^2(1-\phi_3^2)(1-\phi_1^2)$



Q=1 **Q=2** **Q=3** **Q=4**

Baby Skyrme model: solitons

- **Weakly bounding potential** $U(\phi) = \mu^2 \left(\alpha(1-\phi_3) + (1-\alpha)(1-\phi_3)^4 \right)$

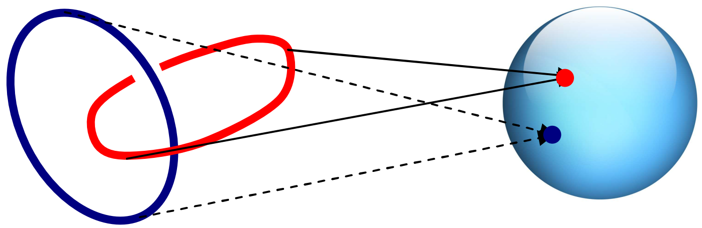


Faddeev-Skyrme model: Construction of the Hopfion

$$L = \frac{1}{2}(\partial_\mu \phi^a)^2 - \frac{\kappa^2}{4}\left(\varepsilon_{abc}\ \phi^a \partial_\mu \phi^b \partial_\nu \phi^c\right)^2$$

Loops in domain space S^3

$\vec{\phi} : S^3 \to S^2$

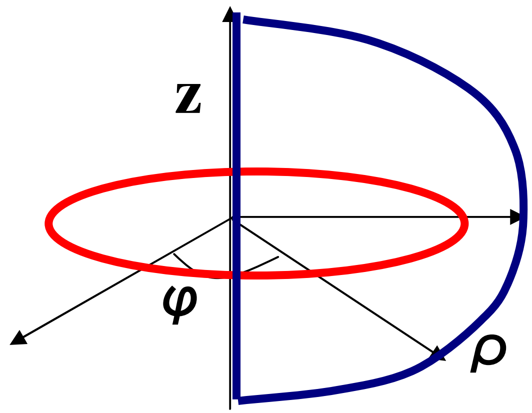


Target space S^2

$$\phi_1 + i\phi^2 = \sin F(\rho, z) e^{in\varphi + iG(\rho,z)z}; \qquad \phi^3 = \cos F(\rho, z)$$

$$F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{F} = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu; \qquad d\mathcal{F} = 0; \qquad \mathcal{F} = d\mathcal{A}$$

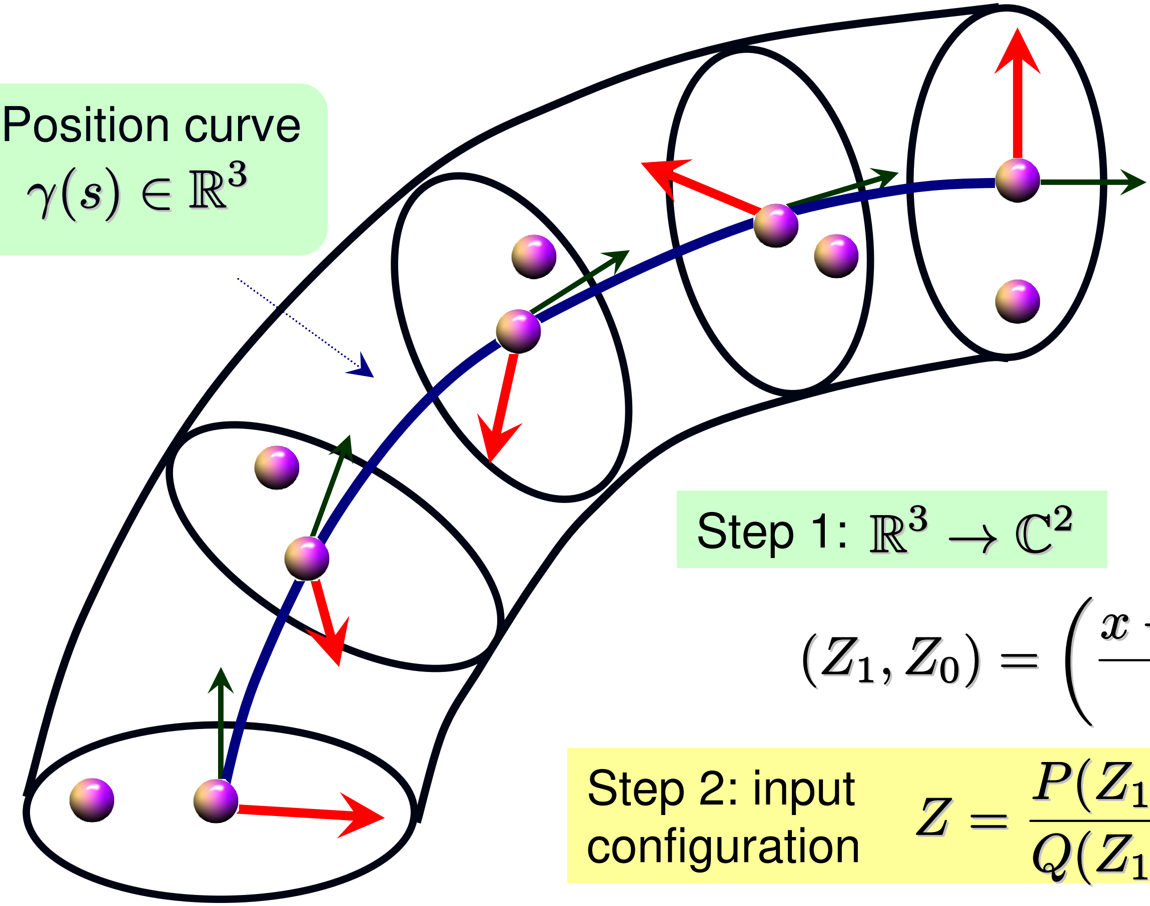


- ***Topological charge:*** $Q = \frac{1}{8\pi^2}\int \varepsilon_{ijk} A_i F_{jk} d^3x$
  (Linking number)

Hopfion from baby Skyrmions

$$\mathcal{A} = n\cos^2\frac{F}{2}dG + m\sin^2\frac{F}{2}d\varphi \qquad \mathcal{A}\wedge\mathcal{F} = nm\cos^2\frac{F}{2}dF\wedge dG\wedge d\varphi$$

Position curve
$\gamma(s) \in \mathbb{R}^3$



$$\boxed{Q = nm}$$

Rational map: $Z : S^3 \to \mathbb{CP}^1$

$$Z = \frac{\phi_1 + i\phi_2}{1+\phi^3}$$

Step 1: $\mathbb{R}^3 \to \mathbb{C}^2$

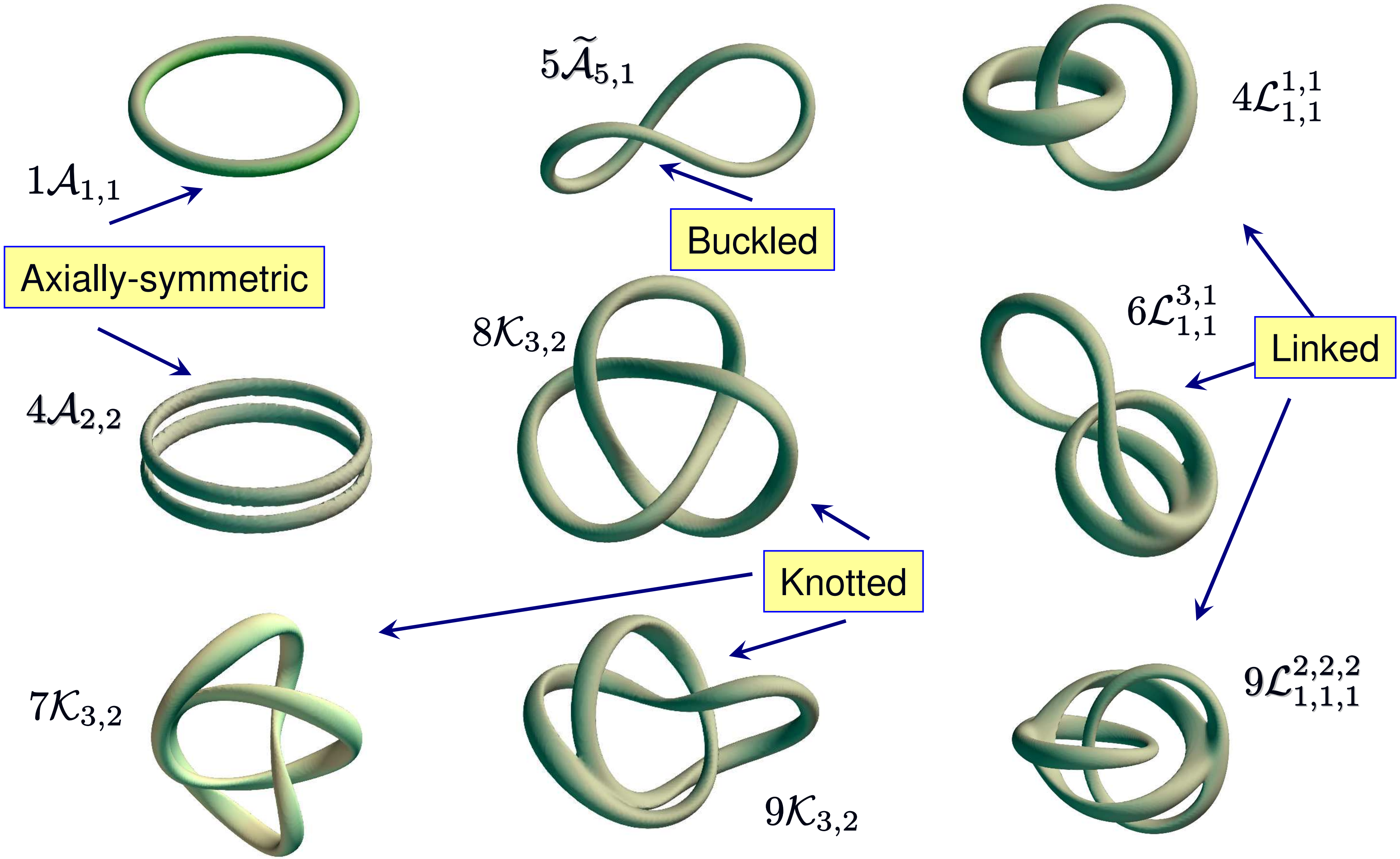
$$(Z_1, Z_0) = \left(\frac{x+ix_2}{r}\sin f, \cos f + i\frac{\sin f}{r}x_3\right)$$

Step 2: input configuration $\quad Z = \dfrac{P(Z_1, Z_0)}{Q(Z_1, Z_0)}$

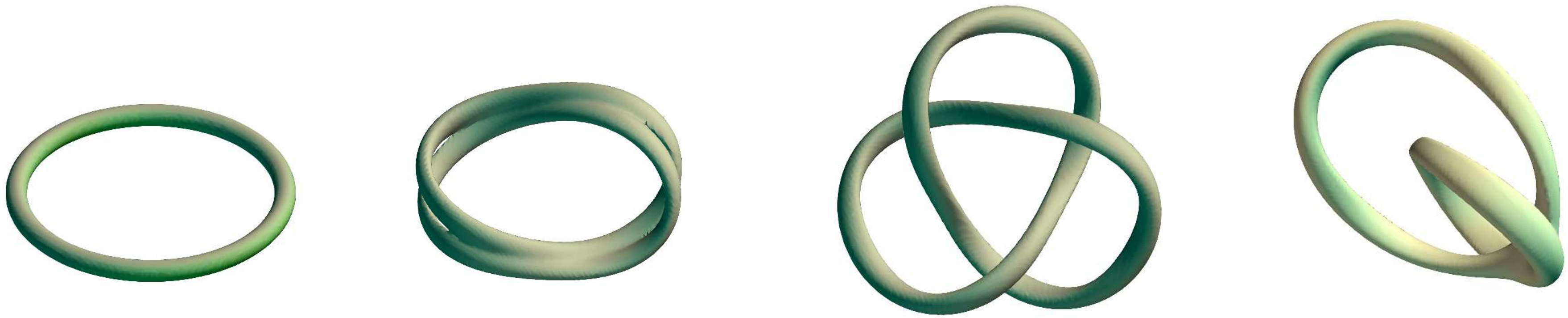
Axially symmetric hopfion $Q\mathcal{A}_{n,m}$

$$\boxed{Z = \frac{Z_1^n}{Z_0^m}}$$

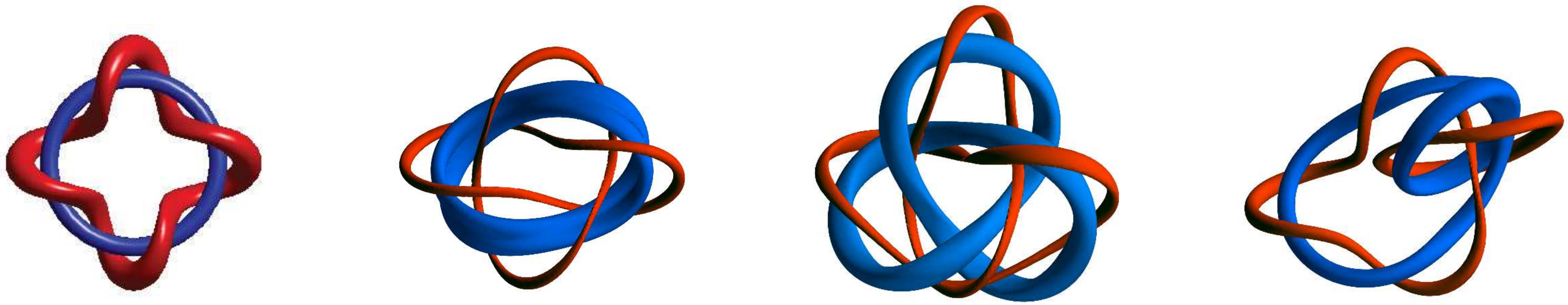
Buckled, linked and knotted hopfions



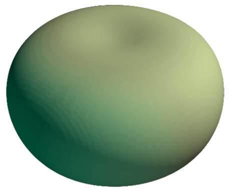
Position curves and linking numbers



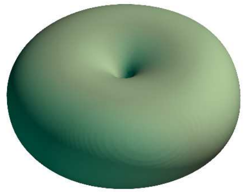
$4\ \mathcal{A}_{4,1}$ $\quad$ $8\widetilde{\mathcal{A}}_{4,2}$ $\quad$ $8\mathcal{K}_{3,2}$ $\quad$ $6\mathcal{L}^{1,1}_{3,1}$



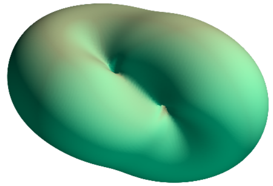
Solitons of the Faddeev-Skyrme model



Q=1 $1\mathcal{A}_{1,1}$

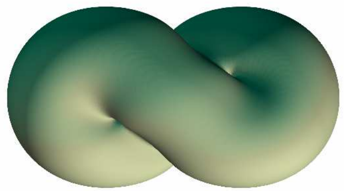


Q=2 $2\mathcal{A}_{2,1}$

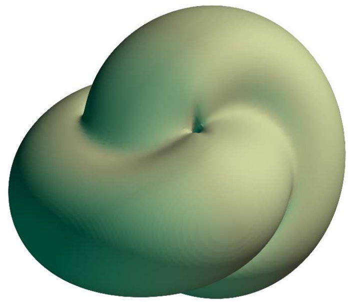


Q=3 $3\widetilde{\mathcal{A}}_{3,1}$

Q=4 $4\mathcal{A}_{2,2}$

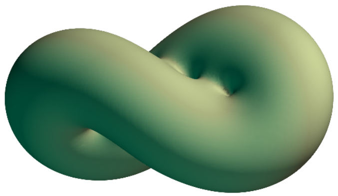


Q=4 $4\widetilde{\mathcal{A}}_{4,1}$

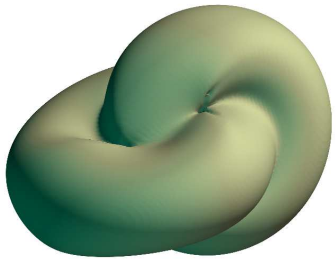


Q=4 $4\mathcal{L}^{1,1}_{1,1}$

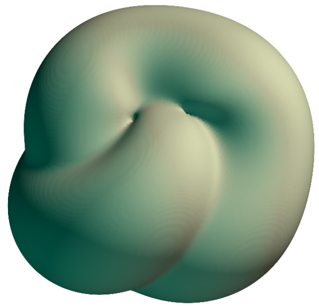
Solitons of the Faddeev-Skyrme model



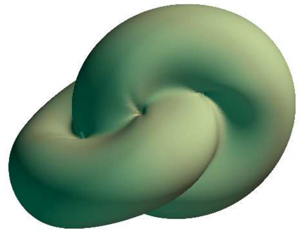
Q=5 $5\widetilde{\mathcal{A}}_{5,1}$



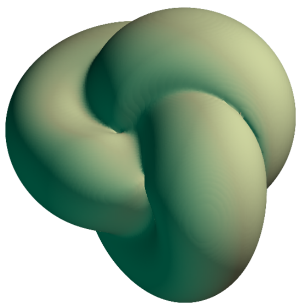
Q=5 $5\mathcal{L}_{1,1}^{1,2}$



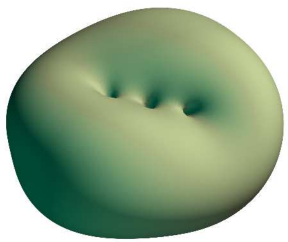
Q=6 $6\mathcal{L}_{1,1}^{2,2}$



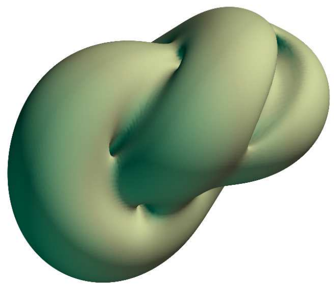
Q=6 $6\mathcal{L}_{1,1}^{3,1}$



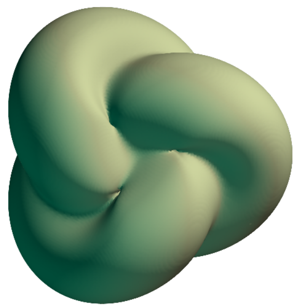
Q=7 $7\mathcal{K}_{3,2}$



Q=8 $8\widetilde{\mathcal{A}}_{4,2}$



Q=8 $8\mathcal{L}_{1,1}^{3,3}$



Q=8 $8\mathcal{K}_{3,2}$

Hopfions with symmetry breaking potential

$$E = \frac{1}{2}(\partial_i \phi^a)^2 + \frac{1}{4}\left(\varepsilon_{abc}\, \phi^a \partial_i \phi^b \partial_j \phi^c\right)^2 + V(\phi)$$

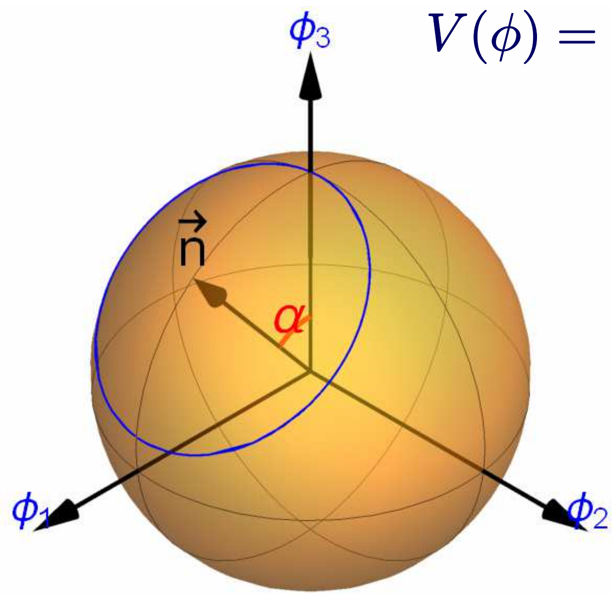
$$\boxed{V(\phi) = \mu^2 \left[\phi_1 \sin\alpha - (1-\phi_3)\cos\alpha\right]^2}$$

$\alpha = 0$

$$V(\phi) = \mu^2 (1-\phi_3)^2$$

$\alpha = \pi/2$

$$V(\phi) = \mu^2 \phi_1^2$$



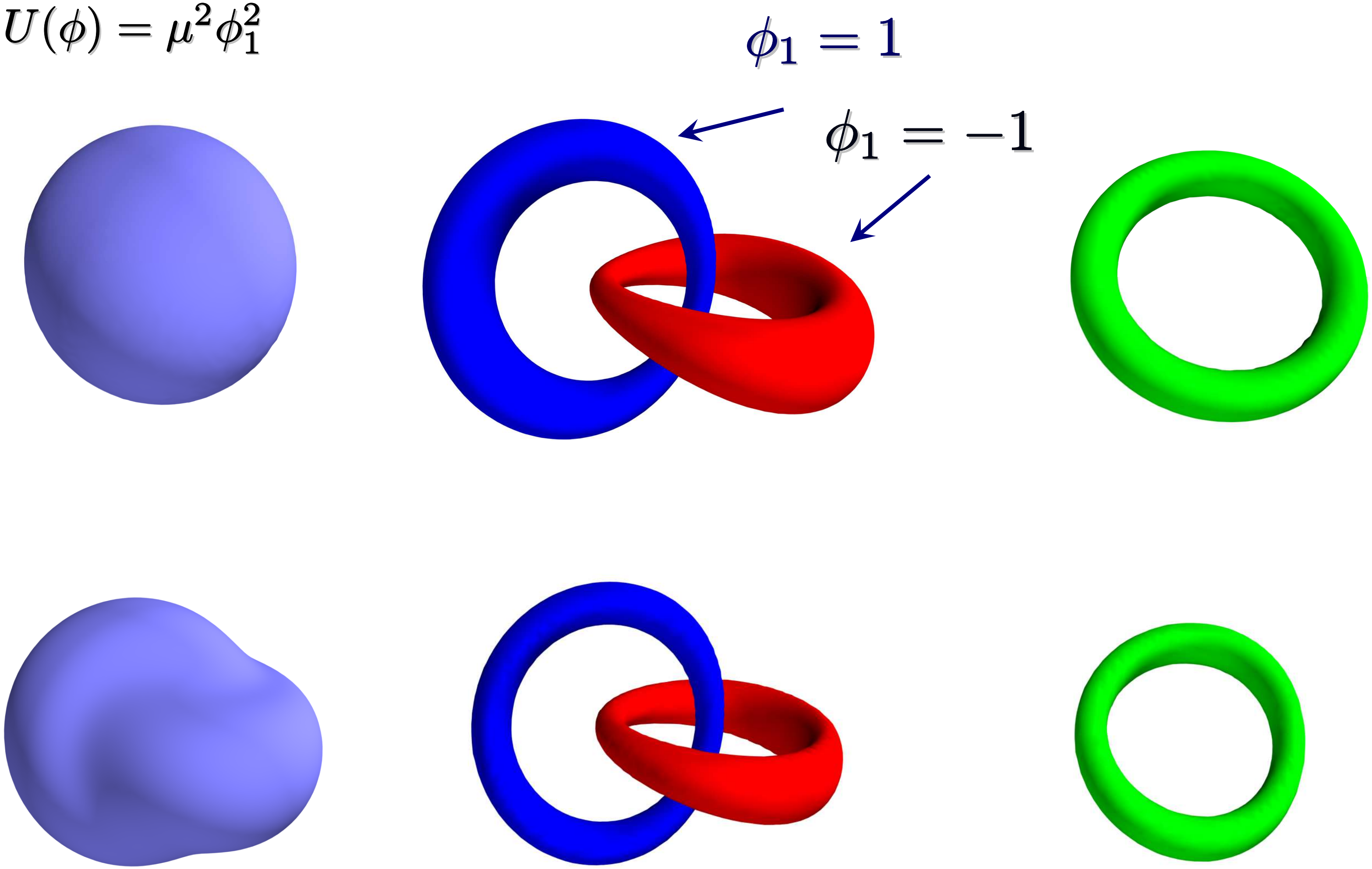
Vacuum boundary conditions: $V(\phi) \to 0$ as $r \to \infty$

Q=1 Hopfion (easy plane potential)

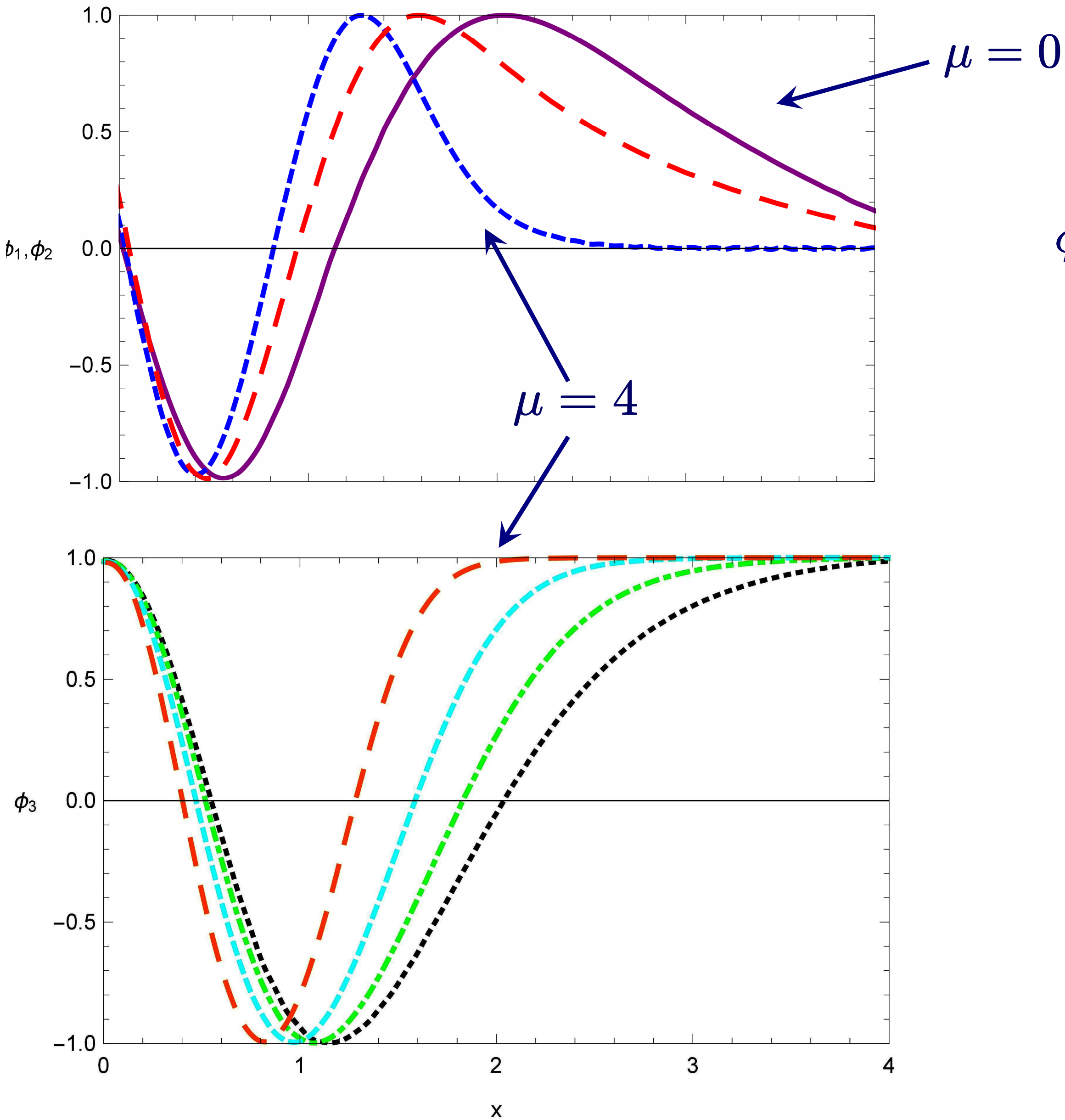
$U(\phi) = \mu^2 \phi_1^2$

μ=0

μ=4



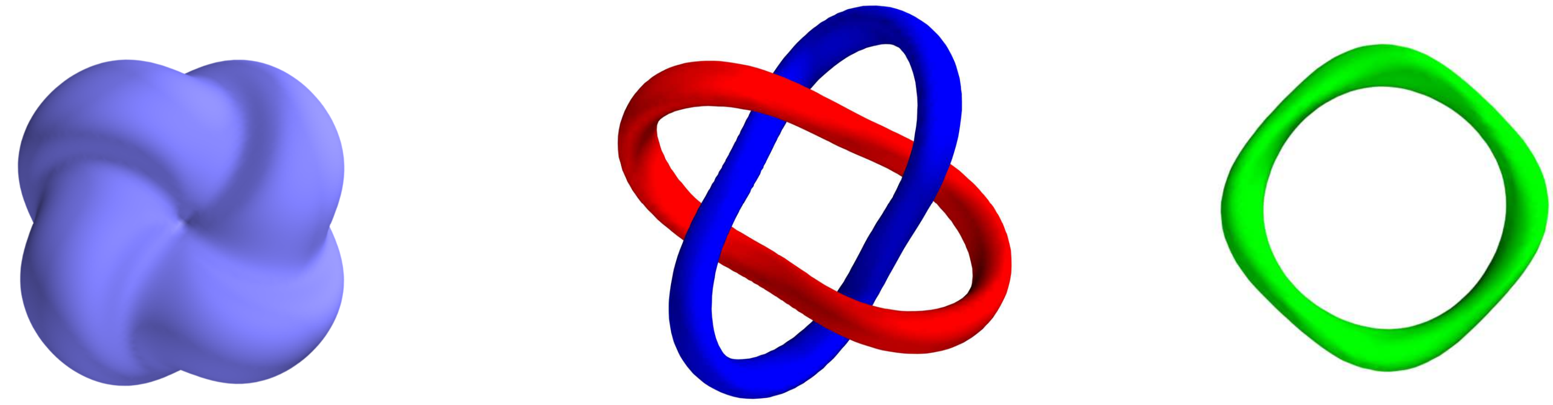
Q=1 Hopfion (easy plane potential)



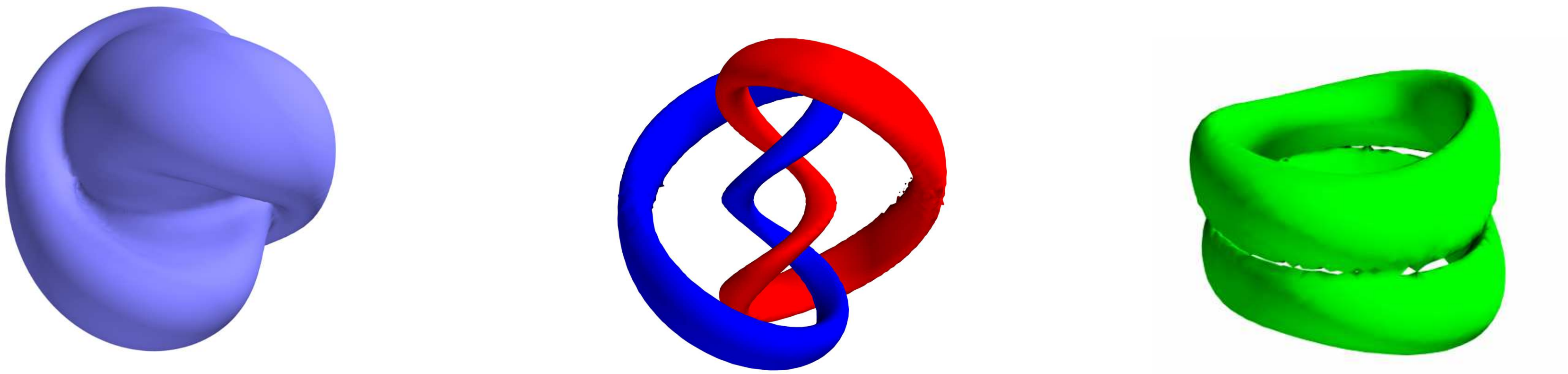
$$\phi_1 \sim \frac{1}{r} e^{-\mu r}, \qquad \phi_2 \sim \frac{1}{r}$$

$$\phi_3 \sim \frac{1}{r} e^{-\mu r}$$

Q=2

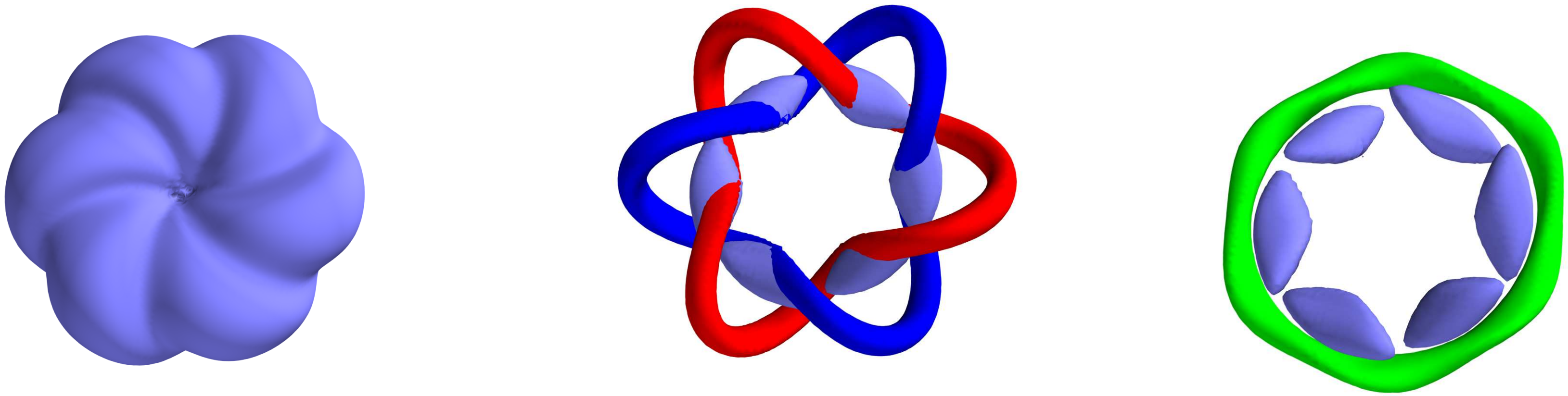


$$2(\widetilde{\mathcal{A}}_1 \between \widetilde{\mathcal{A}}_1)_{\mathcal{A}_{2,1}}$$

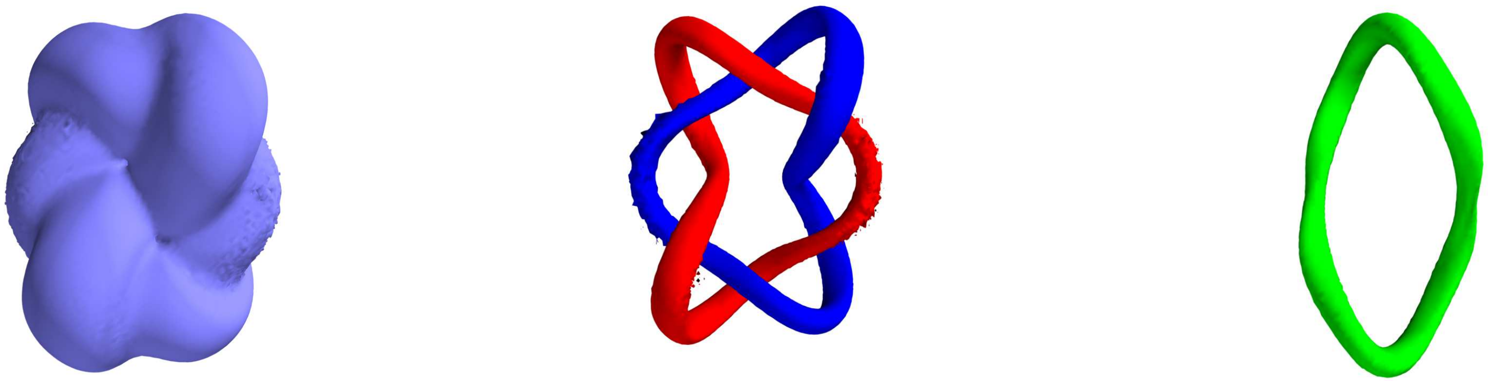


$$2(\widetilde{\mathcal{A}}_1 \between \widetilde{\mathcal{A}}_1)_{\mathcal{A}_{1,2}}$$

Q=3



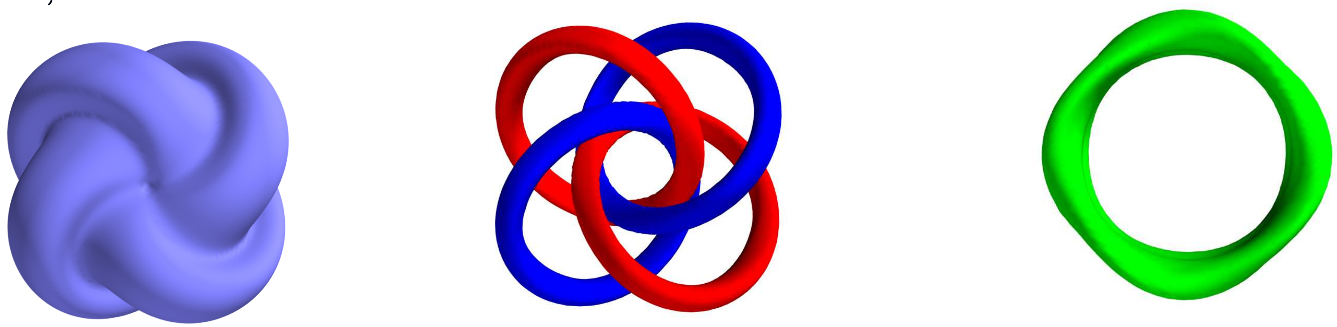
$$3(\widetilde{\mathcal{A}}_1 \between \widetilde{\mathcal{A}}_1)_{\mathcal{A}_{3,1}}$$



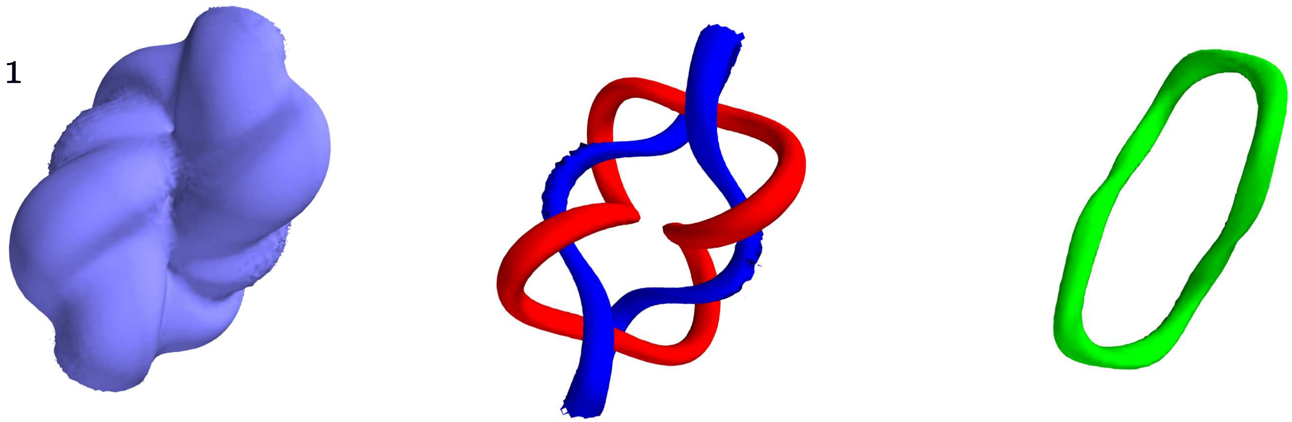
$$3(\widetilde{\mathcal{A}}_1 \between \widetilde{\mathcal{A}}_1)_{\widetilde{\mathcal{A}}_{3,1}}$$

Q=4

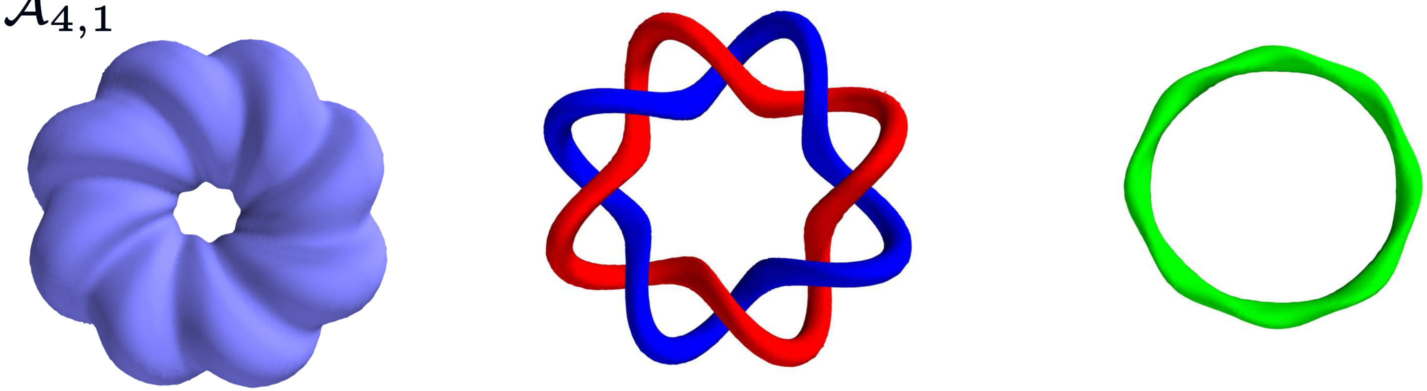
$4(\mathcal{L}_{1,1} \between \mathcal{L}_{1,1})_{\mathcal{A}_{2,2}}$



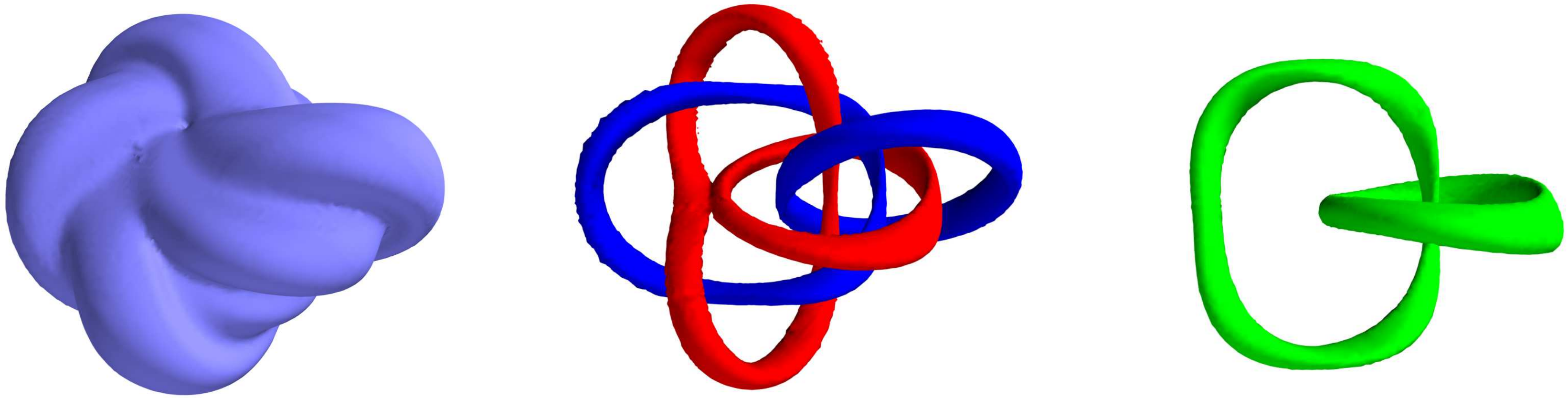
$4(\widetilde{\mathcal{A}}_1 \between \widetilde{\mathcal{A}}_1)_{\widetilde{\mathcal{A}}_{4,1}}$



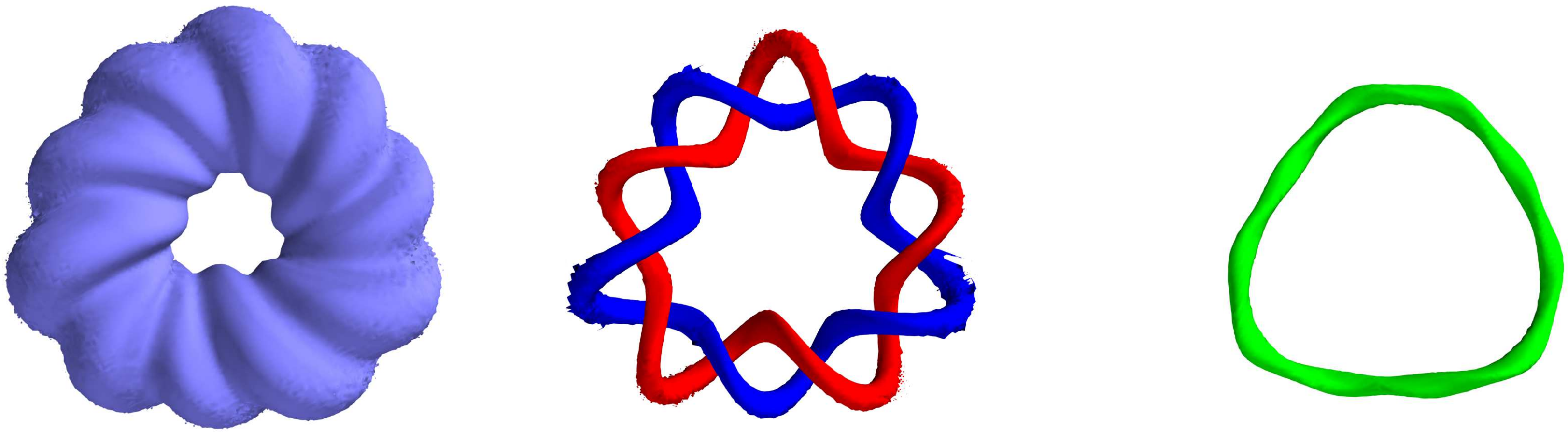
$4(\mathcal{A}_1 \between \mathcal{A}_1)_{\mathcal{A}_{4,1}}$



Q=5



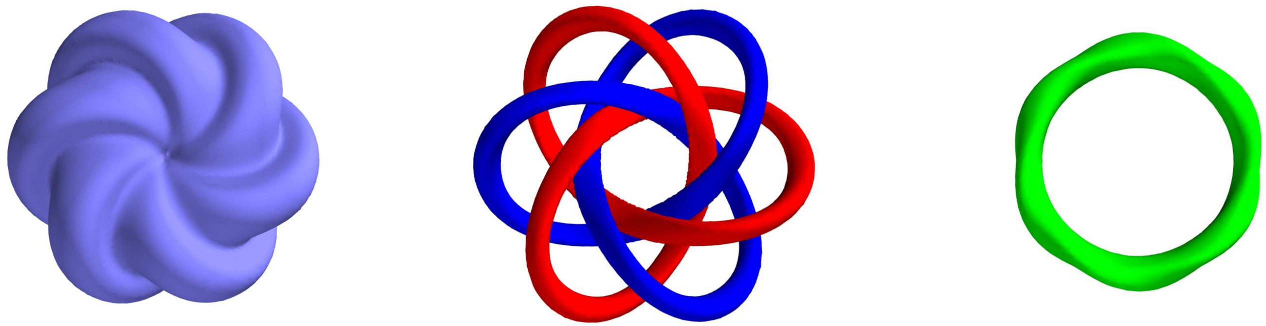
$$5(\mathcal{L}_{1,2} \between \mathcal{L}_{1,2})_{\mathcal{L}_{1,2}}$$



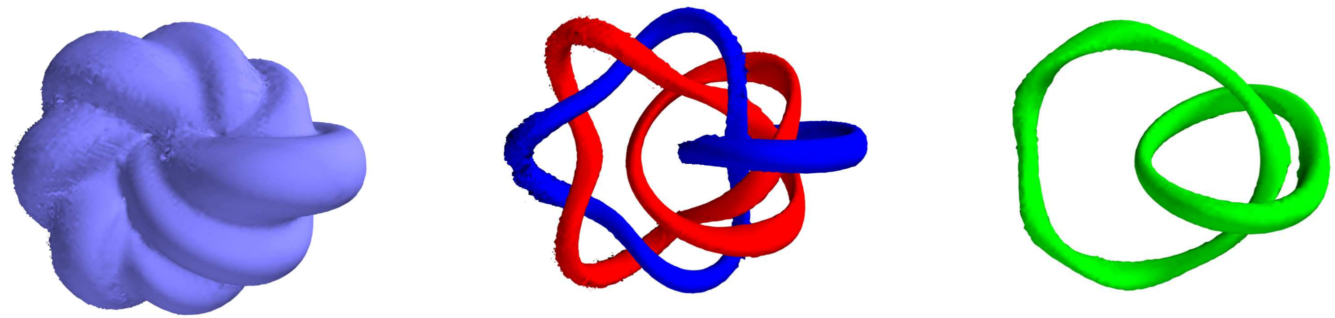
$$5(\mathcal{A}_1 \between \mathcal{A}_1)_{\widetilde{\mathcal{A}}_{5,1}}$$

Q=6

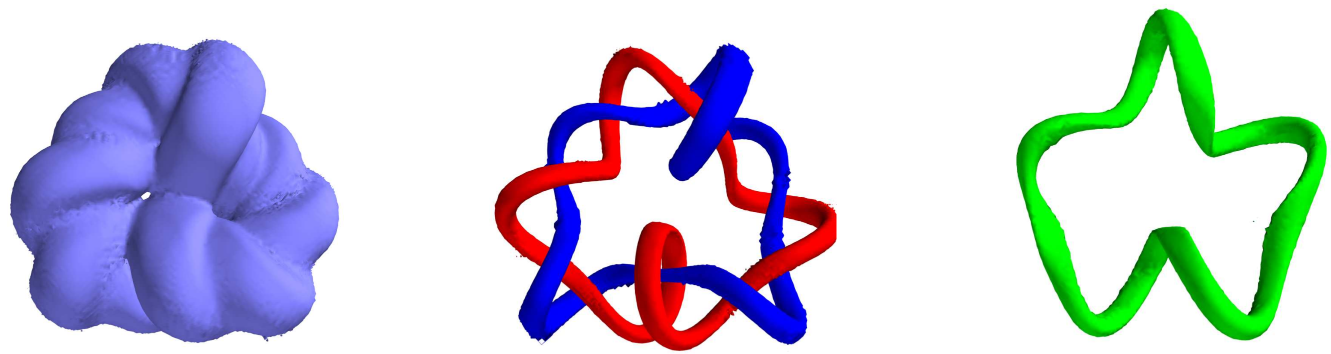
$6(\mathcal{K}_{3,2} \between \mathcal{K}_{3,2})_{\mathcal{A}_{3,2}}$



$6(\mathcal{L}_{1,3} \between \mathcal{K}_{3,2})_{\mathcal{L}^{1,1}_{1,3}}$

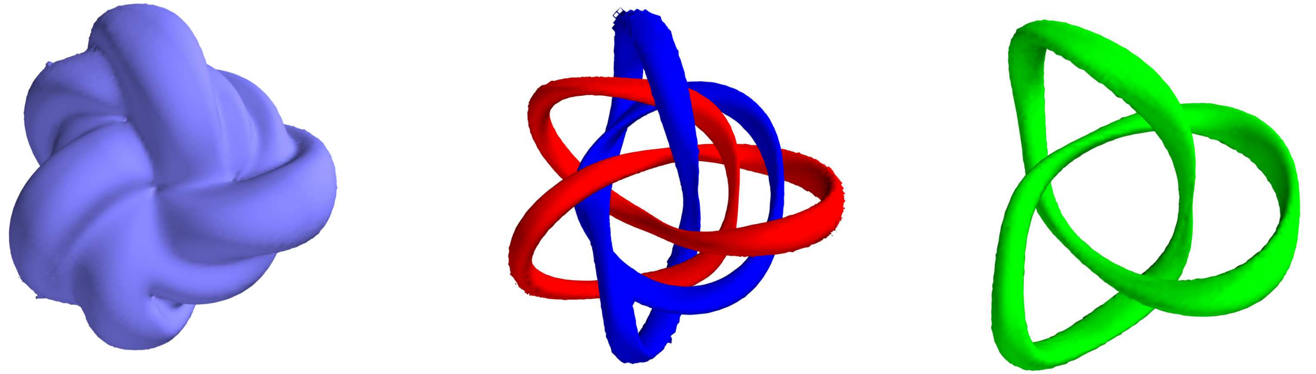


$6(\widetilde{\mathcal{A}}_1 \between \widetilde{\mathcal{A}}_1)_{\widetilde{\mathcal{A}}_{6,1}}$

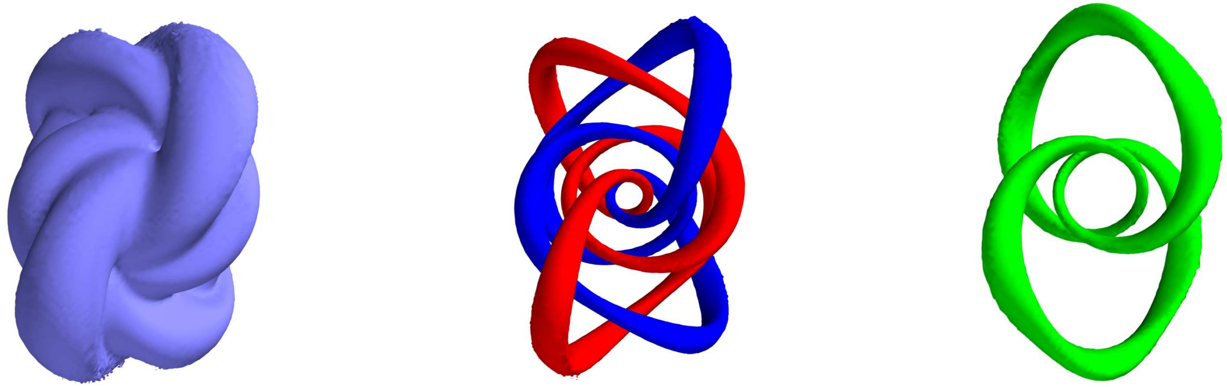


Q=7

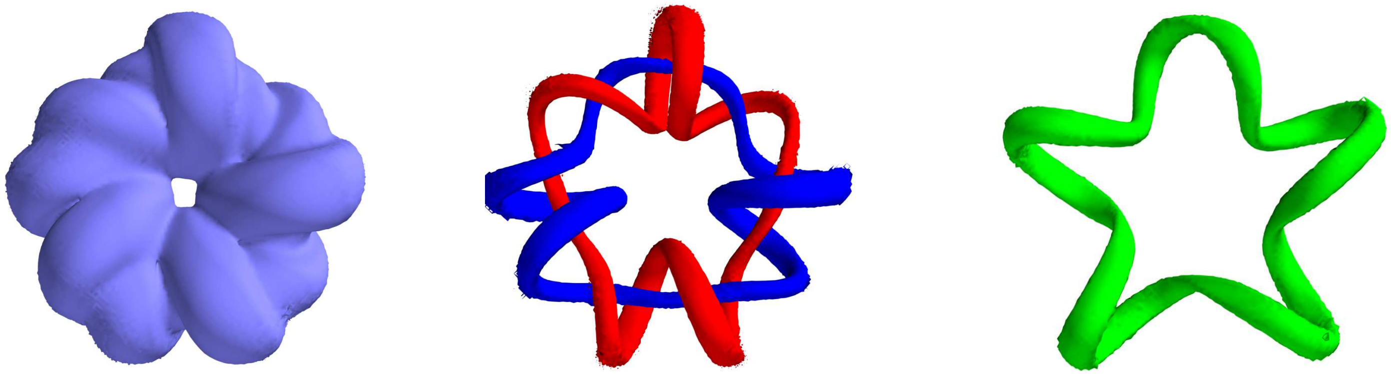
$7(\mathcal{L}^{2,2}_{1,2} \between \mathcal{K}_{3,2})_{\mathcal{K}_{3,2}}$



$7(\mathcal{K}_{2,3} \between \mathcal{K}_{2,3})_{\mathcal{K}_{2,3}}$

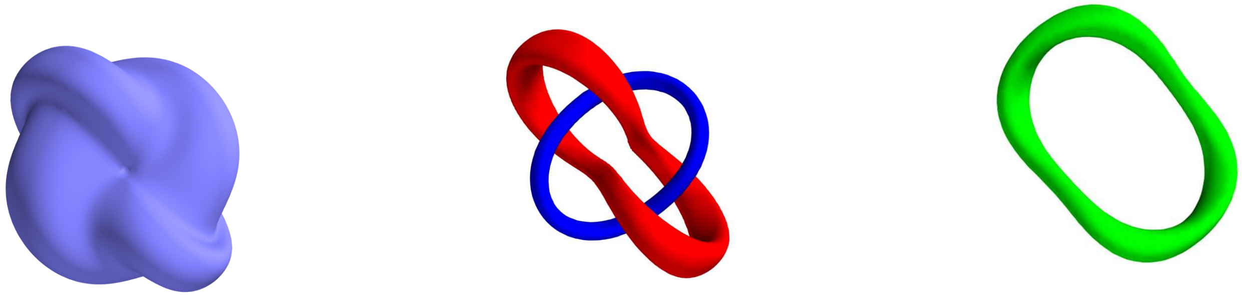


$7(\widetilde{\mathcal{A}}_1 \between \widetilde{\mathcal{A}}_1)_{\widetilde{\mathcal{A}}_{7,1}}$

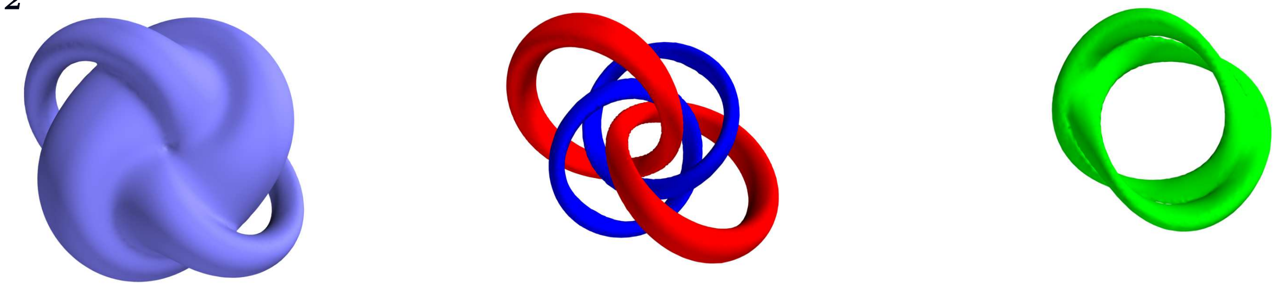


$$\boxed{V(\phi) = \mu^2 \left(\phi_1 - 1/3\right)^2}$$

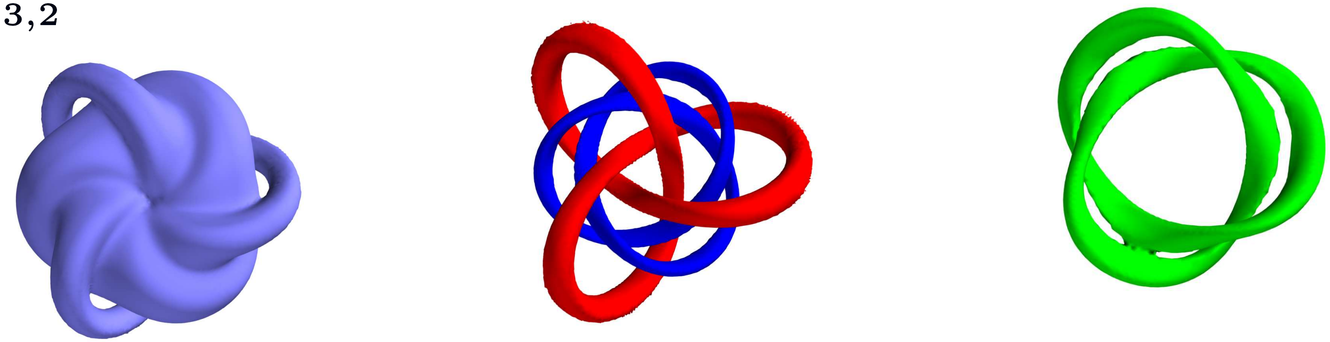
$2(\mathcal{A}_1 \between \widetilde{\mathcal{A}}_1)_{\mathcal{A}_{2,1}}$



$4(\mathcal{L}_{1,1} \between \mathcal{L}_{1,1})_{\mathcal{A}_{2,2}}$



$6(\mathcal{K}_{3,2} \between \mathcal{K}_{3,2})_{\mathcal{K}_{3,2}}$



Gauged Faddeev-Skyrme model

$$\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} - \frac{1}{4}\left(D_\mu\vec{\phi}\times D_\nu\vec{\phi}\right)^2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \qquad D_\mu\vec{\phi} = \partial_\mu\vec{\phi} + gA_\mu\vec{\phi}\times\vec{n}$$

$\phi : S^3 \to S^2; \quad \vec{n} = (0,0,1)$ → SO(2) ≃ U(1) unbroken symmetry group

$$\boxed{(\phi_1 + i\phi_2) = \phi_\perp \to \phi'_\perp = U\phi_\perp; \quad U = e^{ig\alpha} \qquad A_\mu \to A'_\mu = A_\mu + \frac{i}{g}U\partial_\mu U^{-1}}$$

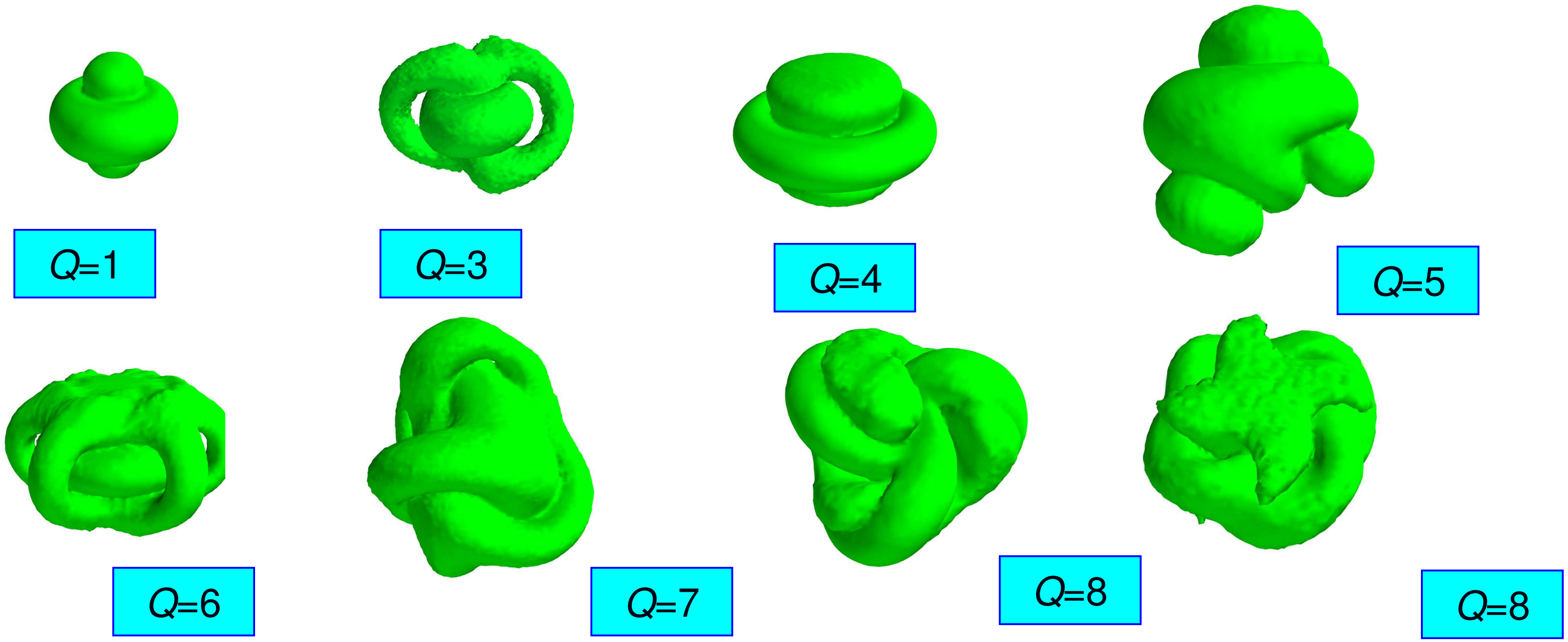
- ***Field equations:***

$$D_\mu\vec{J}^\mu = 0$$

$$\partial_\mu F^{\mu\nu} = 2g\vec{n}\cdot\vec{J}^\nu$$

- ***Current:*** $\vec{J}^\mu = \vec{\phi}\times D^\mu\vec{\phi} - D_\nu\vec{\phi}(D^\nu\vec{\phi}\cdot\vec{\phi}\times D^\mu\vec{\phi})$

Magnetic field: knots and links



Magnetic fluxes follow the curves of $\mathcal{C}_{\pm} = \phi^{-1}(0, 0, \pm 1)$

Summary and Outlook

- Explicit form of the potential strongly affects structure of the multisoltion solutions
- Symmetry breaking potentials yield fractionally charged solitons of the Skyrme-like models
- Magnetic Hopfions have different structure than the usual solitons of the Faddeev-Skyrme model
- In the strong coupling limit the magnetic field of the gauged Hopfions is quantized.
- Gauged solitons with symmetry breaking potential?
- Dynamics of the solitons in the models with symmetry breaking?

Thank you!

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Quantum mechanics could dramatically improve d capacities

Graeme Bu
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By Chris Mellor 5 Aug 2016 at 14:51

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Aug 9, 2013

Skyrmion spin control cou electronics

Researchers at the University o succeeded in controlling tiny m as "skyrmions" for the first time important for future high-densi and nanodigital electronic devic transfer speeds and processing

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Research

Nanosize magnetic whirlpools could be the future of data storage

(2 August 2016)

Could magnetic skyrmions hold the answer to better data storage?

The use of nanoscale magnetic whirlpools, known as magnetic skyrmions, to create novel and efficient ways to store data will be explored in a new £7M research programme led by Durham University.

Skyrmions, which are a new quantum mechanical state of matter, could be used to make our day-to-day gadgets, such as mobile phones and laptops, much smaller and cheaper whilst using less energy and generating less heat.

It is hoped better and more in-depth knowledge of skyrmions could address society's ever-increasing demands for processing and storing large amounts of data and improve current hard drive technology.

Revolutionise data storage

Scientists first predicted the existence of skyrmions in 1962 but they were only discovered experimentally in magnetic materials in 2009.

The UK team, funded by the Engineering and Physical Sciences Research Council (EPSRC), now aims to make a step change in our understanding of skyrmions with the goal of producing a new type of demonstrator device in partnership with industry.

Skyrmions, tiny swirling patterns in magnetic fields, can be created, manipulated and controlled in certain magnetic materials. Inside a skyrmion, magnetic moments point in different directions in a self-organised vortex.