Momentum space topology and anomalous transport phenomena

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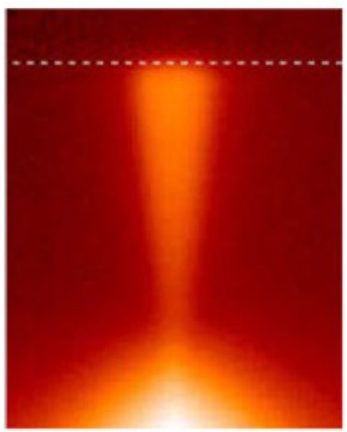
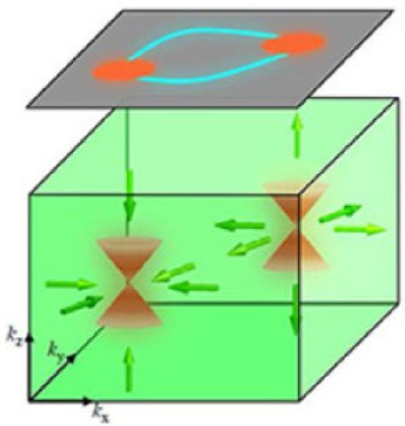
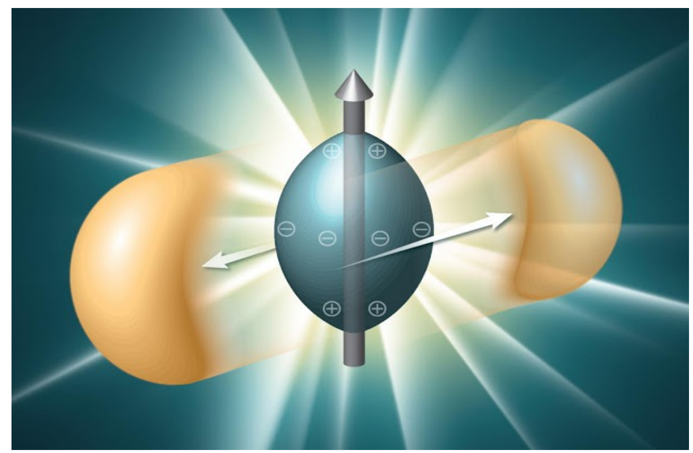
Relations between the high energy physics and the solid state physics







F.A.Berezin

(Wigner transform, deformational quantization, and all that)

G.E.Volovik

(Momentum space topology in condensed matter and beyond)

Unusual analytical

Methods of lattice

Quantum field theory

Condensed

Matter Physics

Momentum space topology

Solid state physics (topological insulators, Weyl semimetals)

High energy physics

Wigner transformation on the lattice,
Momentum space topology

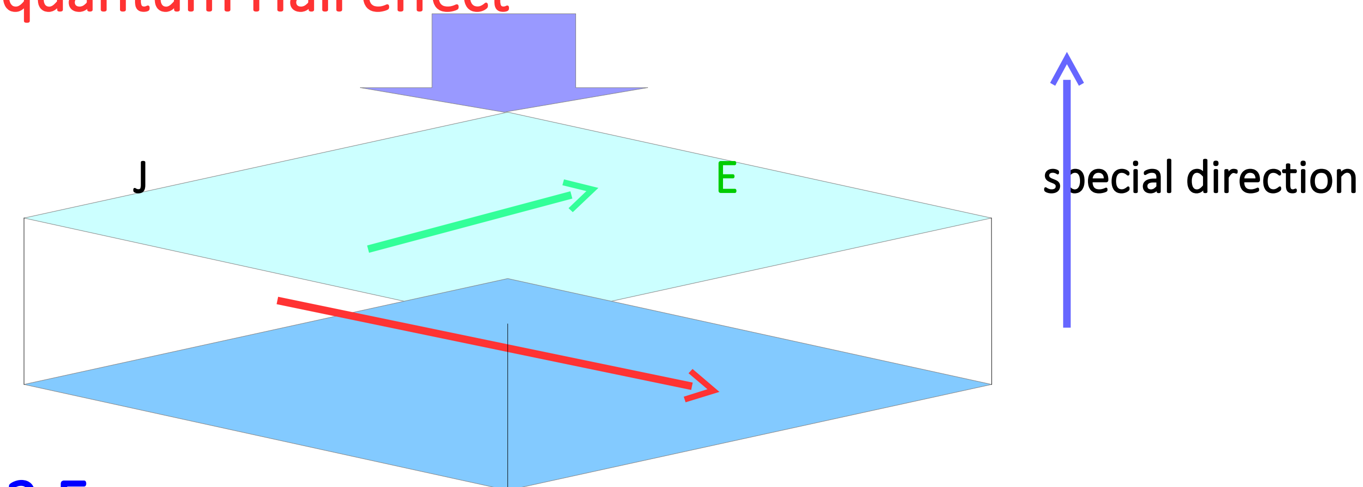
Anomalous Quantum Hall Effect,
(the absence of) Chiral Magnetic Effect,
Chiral Separation effect

Analytical methods in lattice field theory have been applied both to the relativistic QFT and to the solid state physics

Topological insulators, Weyl semimetals, lattice regularized relativistic quantum field theory

Nondissipative current = Topological invariant × Electric or magnetic field

Anomalous quantum Hall effect



$J = M / 4\pi^2 E$

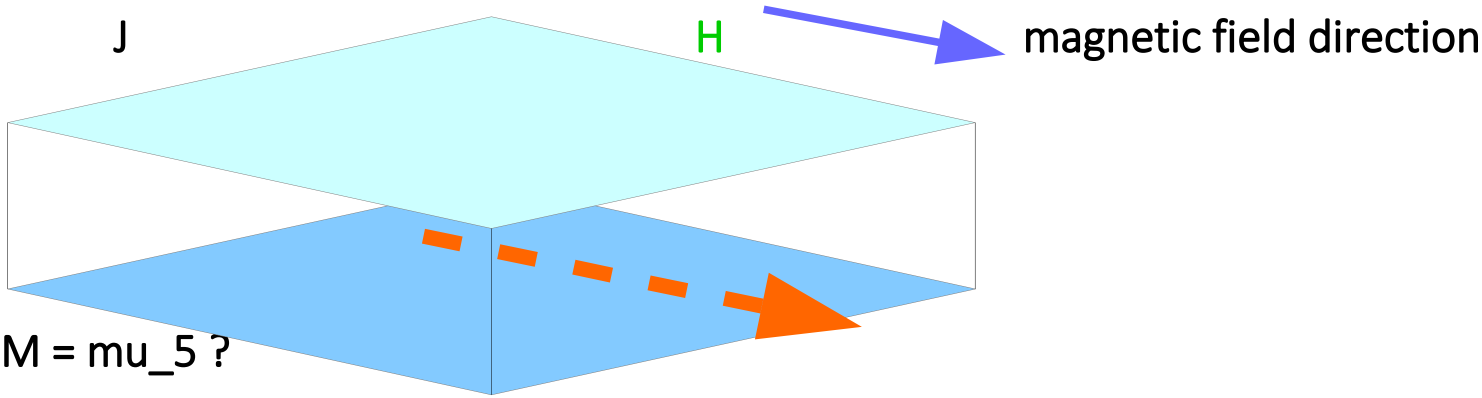
Weyl semimetals M = [distance between the Weyl points in momentum space]

Topological insulators M = const/a = integer x [vector of inverse lattice]

Topological insulators, Weyl semimetals, lattice regularized relativistic quantum field theory

Nondissipative current = Topological invariant × Electric or magnetic field

Chiral magnetic effect



J = M /2pi^2 H   M = mu_5 ?

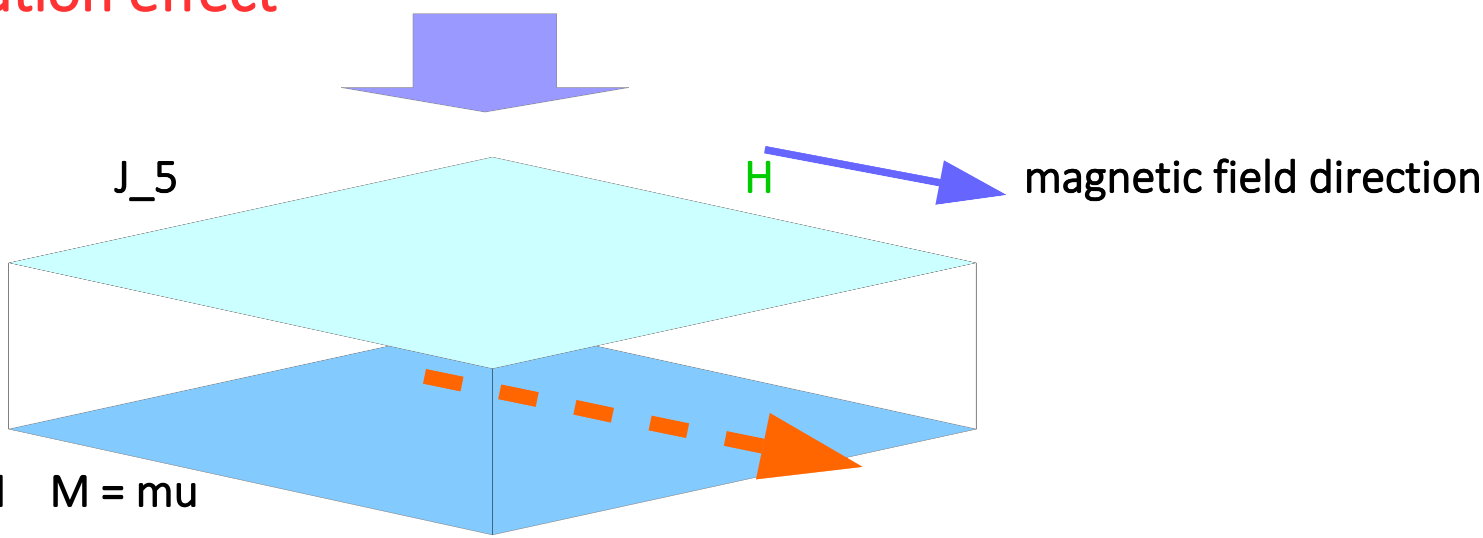
Dirac semimetals

massless relativistc fermions (heavy ion collisions)

Topological insulators, Weyl semimetals, lattice regularized relativistic quantum field theory

Nondissipative current = Topological invariant × Electric or magnetic field

Chiral separation effect



J_5 = M /2pi^2 H   M = mu

Dirac semimetals

massless relativistc fermions (heavy ion collisions)

M.A.Zubkov, « Absence of equilibrium chiral magnetic effect » arXiv:1605.08724, Physical Review D 93, 105036 (2016)

M.A.Zubkov, « Wigner transformation, momentum space topology, and anomalous transport » arXiv:1603.03665, Annals Phys. 373 (2016) 298-324

Z.V.Khaidukov, M.A.Zubkov, "Chiral Separation effect in lattice regularization" Phys. Rev. D 95 (2017), 074502

Plan

1. An ununsual exercise on the analytical methods in lattice quantum field theory :

Wigner transform in compact momentum space ==>

J = [top.invariant in momentum space]x[field strength]

2. Applications :

- Equilibrium CME does not exist because the corresponding top. Invariant = 0

- AQHE in 2+1 D, 3+1 D topological insulators, and in 3+1 D Weyl semimetals

- Chiral Separation Effect for massless or nearly massless systems

Unusual analytical methods in lattice field theory

1) gauge field in momentum representation as the pseudodifferential operator

2) Wigner transformations

3) Derivative expansion

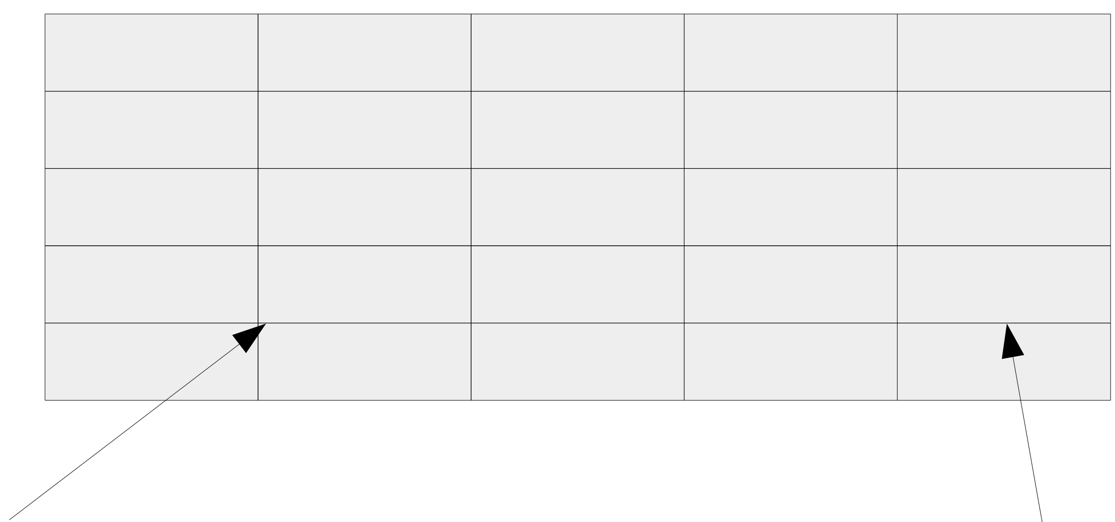
We work with the wide class of lattice models

example

Wilson fermions $Z = \int D\bar{\Psi} D\Psi \exp\Big(- \sum_{\mathbf{r}_n,\mathbf{r}_m} \bar{\Psi}(\mathbf{r}_m)(-i\mathcal{D}_{\mathbf{r}_n,\mathbf{r}_m})\Psi(\mathbf{r}_n)\Big)$

$$\mathcal{D}_{\mathbf{x},\mathbf{y}} = -\frac{1}{2}\sum_i [(1+\gamma^i)\delta_{\mathbf{x}+\mathbf{e}_i,\mathbf{y}} e^{iA_{\mathbf{x}+\mathbf{e}_i,\mathbf{y}}} + (1-\gamma^i)\delta_{\mathbf{x}-\mathbf{e}_i,\mathbf{y}} e^{iA_{\mathbf{x}-\mathbf{e}_i,\mathbf{y}}}] + (m^{(0)}+4)\delta_{\mathbf{x}\mathbf{y}}$$

The lattice:



Fermions are attached to the sites (points).

Gauge field is attached to the links that connect sites.

the model is defined in momentum space:

$$Z = \int D\bar{\Psi} D\Psi \exp\Big(- \int_{\mathcal{M}} \frac{d^D \mathbf{p}}{|\mathcal{M}|} \bar{\Psi}(\mathbf{p}) \mathcal{G}^{-1}(\mathbf{p}) \Psi(\mathbf{p}) \Big)$$

coordinates are discrete ==> momentum space is compact
(electrons in solids and lattice regularized QFT)
In coordinate space

$$\Psi(\mathbf{r}) = \int_{\mathcal{M}} \frac{d^D \mathbf{p}}{|\mathcal{M}|} e^{i\mathbf{p}\mathbf{r}} \Psi(\mathbf{p})$$

$$Z = \int D\bar{\Psi} D\Psi \exp\Big(- \sum_{\mathbf{r}_n} \bar{\Psi}(\mathbf{r}_n) \Big[\mathcal{G}^{-1}(-i\partial_{\mathbf{r}}) \Psi(\mathbf{r}) \Big]_{\mathbf{r}=\mathbf{r}_n} \Big)$$

Example: Wilson fermions (=simple model of top.insulator)

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) - i m(\mathbf{p}) \Big)^{-1}$$

$$g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4} (1 - \cos p_a)$$

How to introduce the gauge field

$$Z = \int D\bar{\Psi} D\Psi \exp\Big(- \int_{\mathcal{M}} \frac{d^D \mathbf{p}}{|\mathcal{M}|} \bar{\Psi}(\mathbf{p}) \mathcal{G}^{-1}(\mathbf{p}) \Psi(\mathbf{p}) \Big)$$

In momentum space:

$$\hat{\mathcal{Q}} = \mathcal{G}^{-1}(\mathbf{p} - \mathbf{A}(i\partial_{\mathbf{p}}))$$

$$p_{i_1} \ldots p_{i_n} \quad ==> \quad \tfrac{1}{n!} \sum\nolimits_{\text{permutations}} (\hat{p}_{i_1} - A_{i_1}) \ldots (\hat{p}_{i_n} - A_{i_n})$$

$$Z = \int D\bar{\Psi} D\Psi \exp\Big(- \int_{\mathcal{M}} \frac{d^D \mathbf{p}}{|\mathcal{M}|} \bar{\Psi}(\mathbf{p}) \hat{\mathcal{Q}}(i\partial_{\mathbf{p}}, \mathbf{p}) \Psi(\mathbf{p}) \Big)$$

For the Wilson fermions the equivalence is exact. For the other models it is up to the irrelevant terms ~a^2 x field strength

Gauge field appears as the pseudo — differential operator in momentum space.

Wigner transformation in momentum space (lattice models)

Two point
Green function

$$G(\mathbf{p}_1,\mathbf{p}_2) = \frac{1}{Z}\int D\bar{\Psi}D\Psi\,\bar{\Psi}(\mathbf{p}_2)\Psi(\mathbf{p}_1)$$
$$\exp\Big(-\int\frac{d^D\mathbf{p}}{|\mathcal{M}|}\bar{\Psi}(\mathbf{p})\hat{Q}(i\partial_{\mathbf{p}},\mathbf{p})\Psi(\mathbf{p})\Big)$$

Wigner transformation:

$$\tilde{G}(\mathbf{R},\mathbf{p}) = \int\frac{d^D\mathbf{P}}{|\mathcal{M}|}e^{i\mathbf{P}\mathbf{R}}G(\mathbf{p}+\mathbf{P}/2,\mathbf{p}-\mathbf{P}/2)$$

In coordinate space:

$$\tilde{G}(\mathbf{R},\mathbf{p}) = \sum_{\mathbf{r}=\mathbf{r}_n} e^{-i\mathbf{p}\mathbf{r}}G(\mathbf{R}+\mathbf{r}/2,\mathbf{R}-\mathbf{r}/2)$$

$$G(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{Z}\int D\bar{\Psi}D\Psi\,\bar{\Psi}(\mathbf{r}_2)\Psi(\mathbf{r}_1)$$
$$\exp\Big(-\frac{1}{2}\sum_{\mathbf{r}_n}\Big[\bar{\Psi}(\mathbf{r}_n)\Big[\mathcal{G}^{-1}(-i\partial_{\mathbf{r}}$$
$$-\mathbf{A}(\mathbf{r}))\Psi(\mathbf{r})\Big]_{\mathbf{r}=\mathbf{r}_n}+(h.c.)\Big]\Big)$$

Wigner transformation in momentum space

Two point
Green function

$$G(\mathbf{p}_1, \mathbf{p}_2) = \frac{1}{Z} \int D\bar{\Psi} D\Psi \, \bar{\Psi}(\mathbf{p}_2)\Psi(\mathbf{p}_1)$$
$$\exp\Big(- \int \frac{d^D\mathbf{p}}{|\mathcal{M}|} \bar{\Psi}(\mathbf{p})\hat{Q}(i\partial_{\mathbf{p}}, \mathbf{p})\Psi(\mathbf{p})\Big)$$

Wigner transformation:

$$\tilde{G}(\mathbf{R}, \mathbf{p}) = \int \frac{d^D\mathbf{P}}{|\mathcal{M}|} e^{i\mathbf{P}\mathbf{R}} G(\mathbf{p} + \mathbf{P}/2, \mathbf{p} - \mathbf{P}/2)$$

Groenewold equation

$$1 = \mathcal{Q}(\mathbf{R}, \mathbf{p}) * \tilde{G}(\mathbf{R}, \mathbf{p})$$
$$\equiv \mathcal{Q}(\mathbf{R}, \mathbf{p}) e^{\frac{i}{2}(\overleftarrow{\partial}_{\mathbf{R}} \overrightarrow{\partial}_{\mathbf{p}} - \overleftarrow{\partial}_{\mathbf{p}} \overrightarrow{\partial}_{\mathbf{R}})} \tilde{G}(\mathbf{R}, \mathbf{p})$$

Weyl symbol of operator

Wigner transform of
matrix element

$$\mathcal{Q}(\mathbf{R}, \mathbf{p}) = \int d^D\mathbf{K} d^D\mathbf{P} e^{i\mathbf{P}\mathbf{R}} \delta(\mathbf{p} - \mathbf{P}/2 - \mathbf{K})$$
$$\times \hat{\mathcal{Q}}(i\partial_{\mathbf{K}}, \mathbf{K}) \delta(\mathbf{p} + \mathbf{P}/2 - \mathbf{K}).$$

Wigner transformation in momentum space

Two point
Green function

$$G(\mathbf{p}_1,\mathbf{p}_2) = \frac{1}{Z}\int D\bar{\Psi}D\Psi\,\bar{\Psi}(\mathbf{p}_2)\Psi(\mathbf{p}_1)$$
$$\exp\Big(-\int \frac{d^D\mathbf{p}}{|\mathcal{M}|}\bar{\Psi}(\mathbf{p})\hat{Q}(i\partial_{\mathbf{p}},\mathbf{p})\Psi(\mathbf{p})\Big)$$

Wigner transformation:

$$\tilde{G}(\mathbf{R},\mathbf{p}) = \int \frac{d^D\mathbf{P}}{|\mathcal{M}|}e^{i\mathbf{P}\mathbf{R}}G(\mathbf{p}+\mathbf{P}/2,\mathbf{p}-\mathbf{P}/2)$$

Groenewold equation

$$1 = \mathcal{Q}(\mathbf{R},\mathbf{p}) * \tilde{G}(\mathbf{R},\mathbf{p})$$
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Weyl symbol of operator

If $\hat{\mathcal{Q}}(\mathbf{r},\hat{\mathbf{p}}) = \mathcal{G}^{-1}(\mathbf{p}-\mathbf{A}(i\partial_{\mathbf{p}}))$ ==> $\mathcal{Q}(\mathbf{r},\mathbf{p}) = \mathcal{G}^{-1}(\mathbf{p}-\mathbf{A}(\mathbf{r})) + O([\partial_i A_j]^2)$

For Wilson fermions the relation is exact if the field strength is constant

Electric current

$$j^k(\mathbf{R}) \quad = \quad \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{V}||\mathcal{M}|} \, \mathrm{Tr}\, \tilde{G}(\mathbf{R},\mathbf{p}) \frac{\partial}{\partial p_k} \Big[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\Big]^{-1}$$

$$(2\pi)^D$$

$$\delta \log Z = \sum\nolimits_{\mathbf{R}=\mathbf{R}_n} j^k(\mathbf{R}) \delta A_k(\mathbf{R}) |\mathcal{V}|$$

Solution of the Groenewold equation

$$
\begin{aligned}
1 &= \mathcal{Q}(\mathbf{R},\mathbf{p}) * \tilde{G}(\mathbf{R},\mathbf{p}) \\
&\equiv \mathcal{Q}(\mathbf{R},\mathbf{p}) e^{\frac{i}{2}(\overleftarrow{\partial}_{\mathbf{R}}\overrightarrow{\partial}_{\mathbf{p}} - \overleftarrow{\partial}_{\mathbf{p}}\overrightarrow{\partial}_{\mathbf{R}})} \tilde{G}(\mathbf{R},\mathbf{p})
\end{aligned}
$$

Derivative expansion

$$
\begin{aligned}
\tilde{G}(\mathbf{R},\mathbf{p}) &= \tilde{G}^{(0)}(\mathbf{R},\mathbf{p}) + \tilde{G}^{(1)}(\mathbf{R},\mathbf{p}) + \ldots \\
\tilde{G}^{(1)} &= -\frac{i}{2}\tilde{G}^{(0)} \frac{\partial\left[\tilde{G}^{(0)}\right]^{-1}}{\partial p_i} \tilde{G}^{(0)} \frac{\partial\left[\tilde{G}^{(0)}\right]^{-1}}{\partial p_j} \tilde{G}^{(0)} A_{ij}(\mathbf{R})
\end{aligned}
$$

with $\quad \tilde{G}^{(0)}(\mathbf{R},\mathbf{p}) = \mathcal{G}(\mathbf{p} - \mathbf{A}(\mathbf{R})) \qquad \hat{\mathcal{Q}} = \mathcal{G}^{-1}(\mathbf{p} - \mathbf{A}(i\partial_{\mathbf{p}}))$

$$
Z = \int D\bar{\Psi} D\Psi \exp\Big(- \int_{\mathcal{M}} \frac{d^D\mathbf{p}}{|\mathcal{M}|} \bar{\Psi}(\mathbf{p}) \hat{\mathcal{Q}}(i\partial_{\mathbf{p}}, \mathbf{p}) \Psi(\mathbf{p}) \Big)
$$

Response of electric current to the gauge field

$$j^k(\mathbf{R}) = j^{(0)k}(\mathbf{R}) + j^{(1)k}(\mathbf{R}) + \dots$$

$$j^{(0)k}(\mathbf{R}) = \int \frac{d^D\mathbf{p}}{(2\pi)^D}\,\mathrm{Tr}\,\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\frac{\partial\left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]^{-1}}{\partial p_k}$$

with

$$j^{(1)k}(\mathbf{R}) = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l A_{ij}(\mathbf{R}),$$

$$\mathcal{M}_l = \int \mathrm{Tr}\,\nu_l\, d^4p$$

Topological invariant
3+1 D

$$\nu_l = -\frac{i}{3!\,8\pi^2}\,\epsilon_{ijkl}\left[\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_i}\frac{\partial\mathcal{G}}{\partial p_j}\frac{\partial\mathcal{G}^{-1}}{\partial p_k}\right]$$

To have the well — defined expressions we need:

1) Ultraviolet regularization **MASSIVE**

2) Infrared regularization **LATTCE FERMIONS**

Lattice regularized quantum field theory

$$Z = \int D\bar{\Psi} D\Psi \exp\Big(- \int_{\mathcal{M}} \frac{d^D \mathbf{p}}{|\mathcal{M}|} \bar{\Psi}(\mathbf{p}) \hat{\mathcal{Q}}(i\partial_{\mathbf{p}}, \mathbf{p}) \Psi(\mathbf{p}) \Big)$$

gauge field as the pseudodifferential operator in momentum space

$$\hat{\mathcal{Q}} = \mathcal{G}^{-1}(\mathbf{p} - \mathbf{A}(i\partial_{\mathbf{p}}))$$

Wigner transformation of the Green function, Weyl symbols of operators

$$j^k(\mathbf{R}) = \int_{\mathcal{M}} \frac{d^D \mathbf{p}}{|\mathcal{V}||\mathcal{M}|} \mathrm{Tr}\, \tilde{G}(\mathbf{R}, \mathbf{p}) \frac{\partial}{\partial p_k} \Big[\tilde{G}^{(0)}(\mathbf{R}, \mathbf{p}) \Big]^{-1}$$

Derivative expansion. Iterative solution of the Groenewold equation

$$\begin{aligned} 1 &= \mathcal{Q}(\mathbf{R}, \mathbf{p}) * \tilde{G}(\mathbf{R}, \mathbf{p}) \\ &\equiv \mathcal{Q}(\mathbf{R}, \mathbf{p}) e^{\frac{i}{2}(\overleftarrow{\partial}_{\mathbf{R}} \overrightarrow{\partial}_{\mathbf{p}} - \overleftarrow{\partial}_{\mathbf{p}} \overrightarrow{\partial}_{\mathbf{R}})} \tilde{G}(\mathbf{R}, \mathbf{p}) \end{aligned}$$

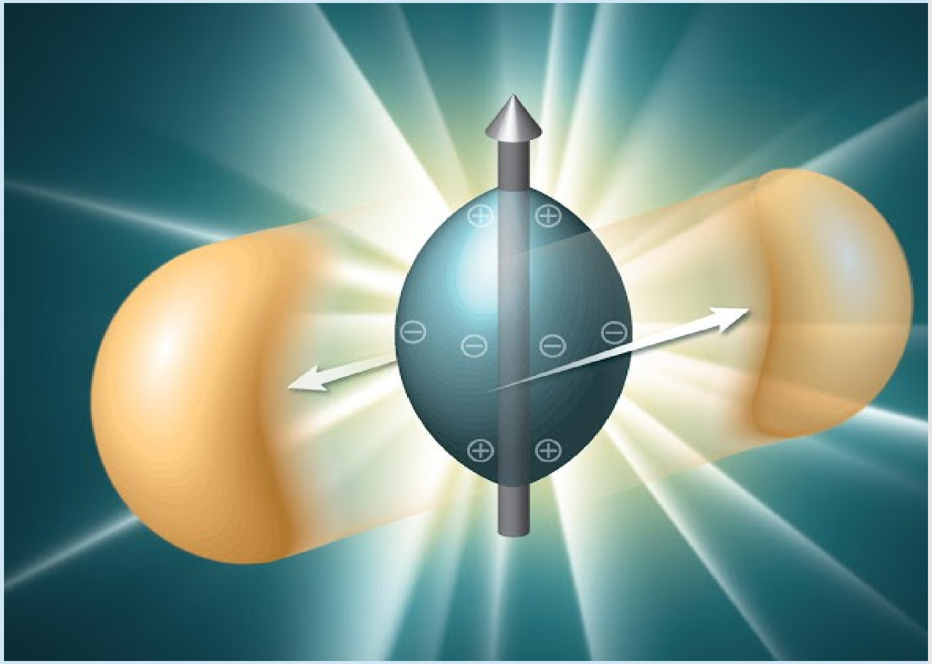
Nondissipative current = Topological invariant × Electric or magnetic field

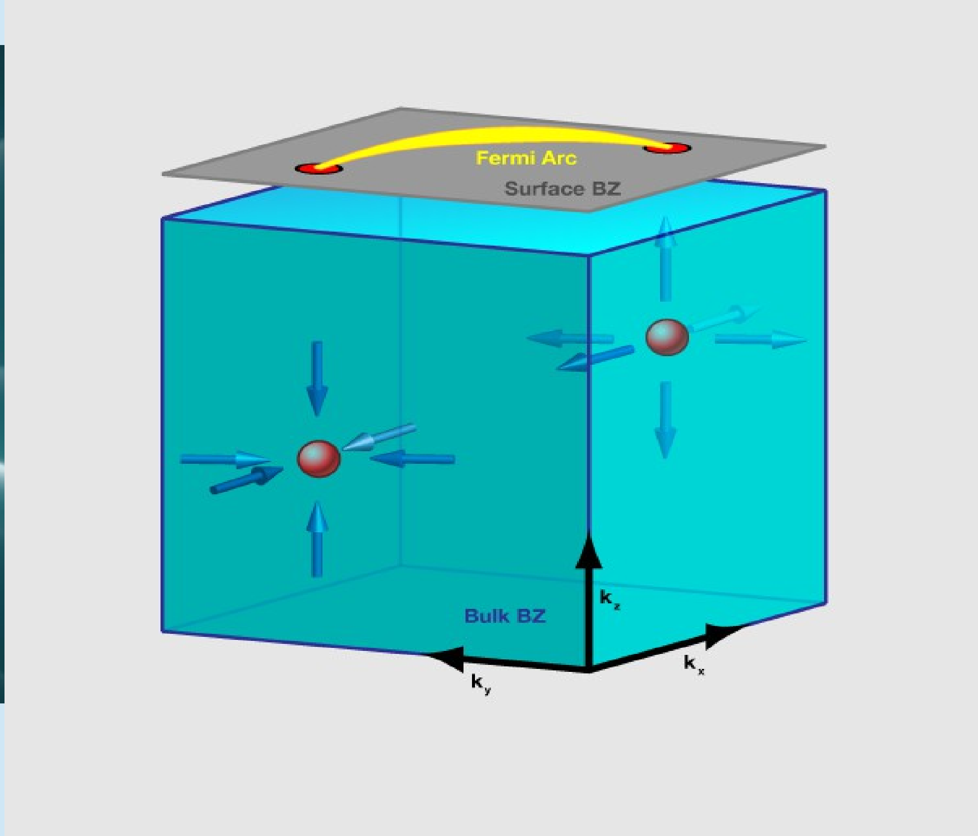
Applications

1. *Equilibrium CME*

M.A.Zubkov, « Absence of equilibrium chiral magnetic effect »
arXiv:1605.08724, Physical Review D 93, 105036 (2016)

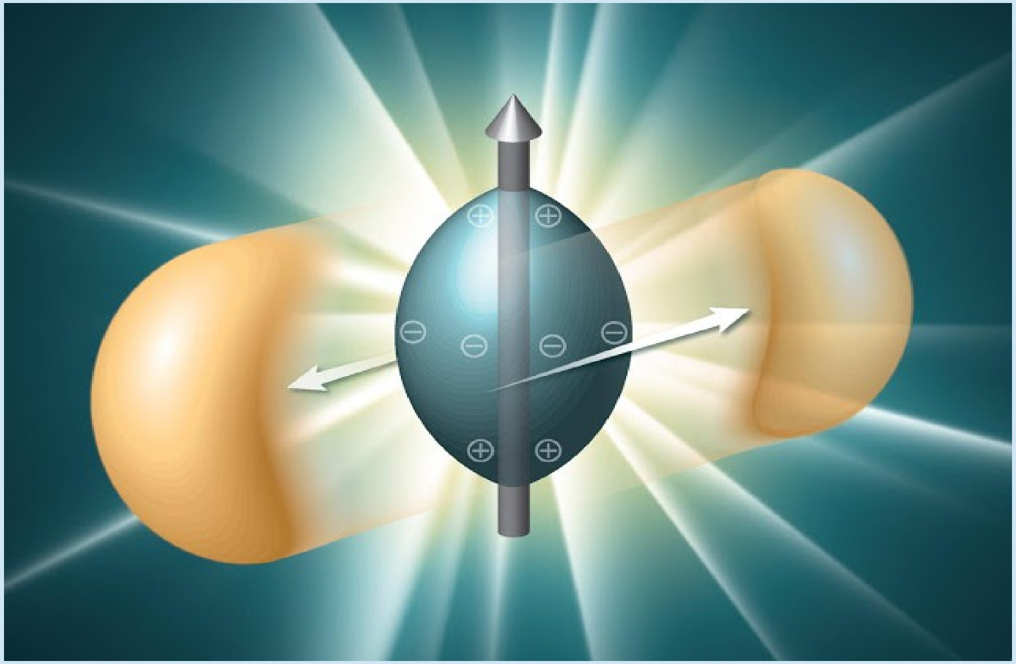
This work argues that the simplest version of this effect does not exist. The presented analytical proof is valid for the wide class of systems both in condensed matter physics and in high energy physics



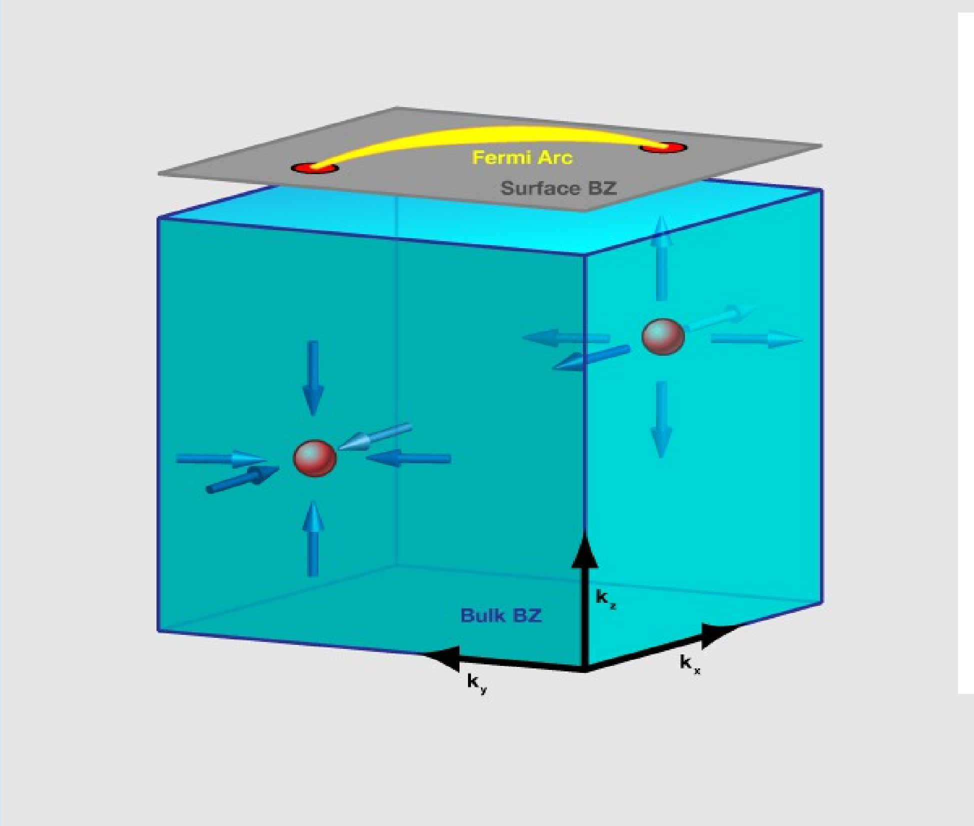


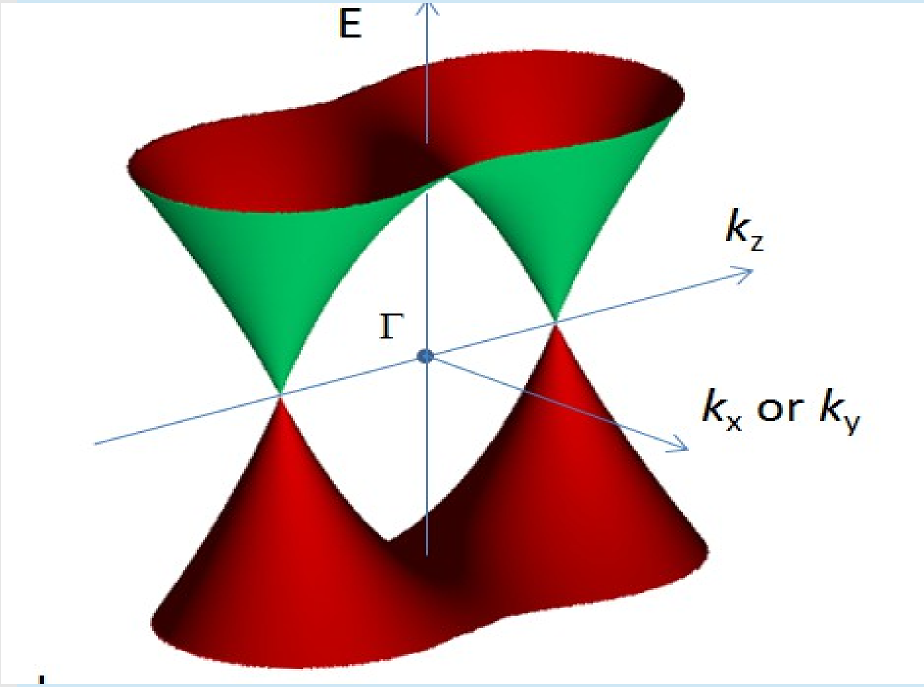
Heavy ion collisions ==> fireball of the quark – gluon plasma ==>

==> massless fermions + magnetic field

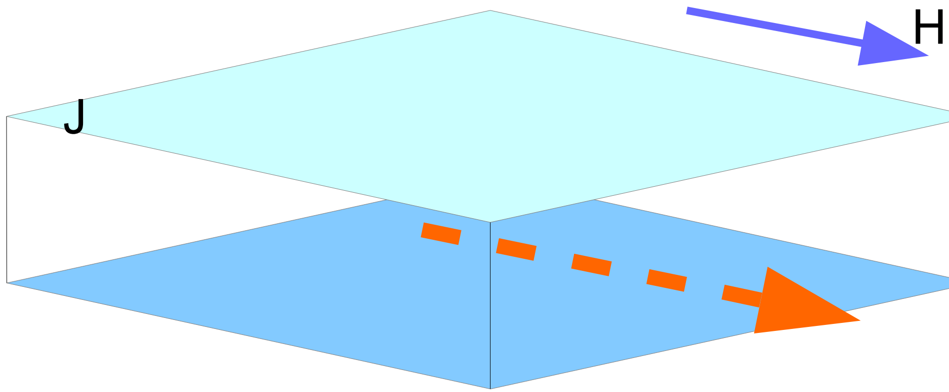


A **Weyl semimetal** is a solid state crystal whose low energy excitations are Weyl fermions. A Weyl semimetal enables the first-ever realization of Weyl fermions. It is a topologically nontrivial phase of matter that broadens the topological classification beyond topological insulators The Weyl semimetal, in which the Weyl points of opposite chiralities coincide is called **Dirac semimetal**





Chiral Magnetic Effect (CME) is the appearance of electric current in the direction of the external magnetic field in the presence of chiral chemical potential



J = M /2pi^2 H   M = mu_5 ?

Pre – history: the existence of chiral magnetic effect was proposed in

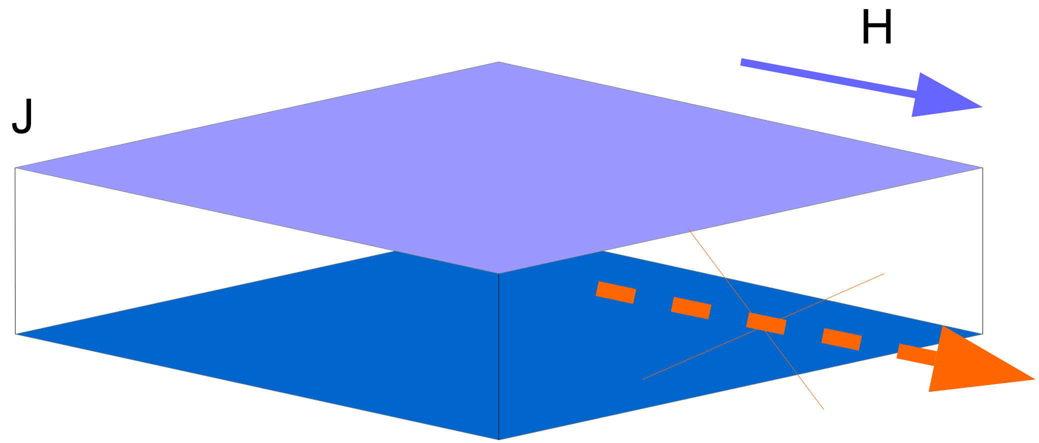
A. Vilenkin, Equilibrium parity-violating current in a magnetic field, Phys. Rev. D 22, 3080 (1980).

This proposition was later repeated in

K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Chiral magnetic effect, Phys. Rev. D 78, 074033 (2008).

and in the sequence of the other papers

Chiral Magnetic Effect (CME) is the appearance of electric current in the direction of the external magnetic field in the presence of chiral chemical potential



Later the existance of the equilibrium bulk static CME was questioned.

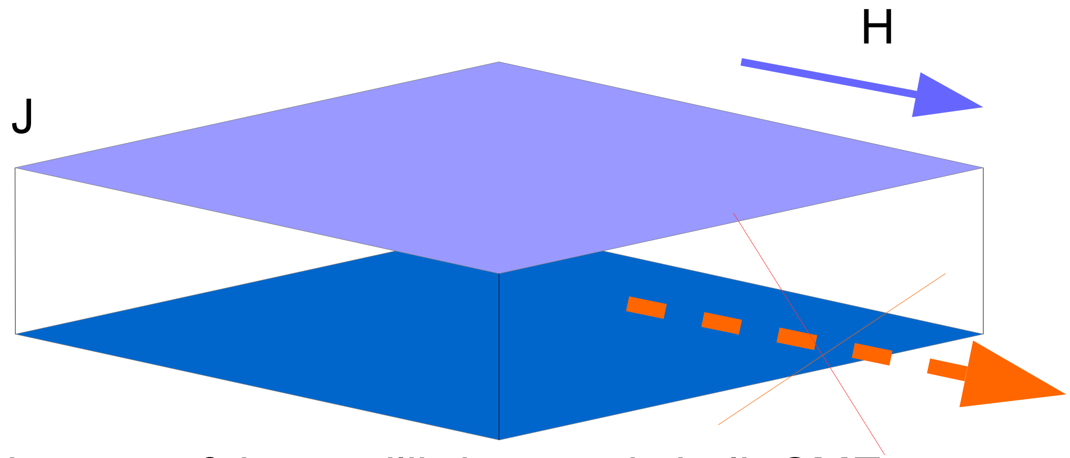
S. N. Valgushev, M. Puhr, and P. V. Buividovich, Chiral magnetic effect in finite-size samples of parity-breaking Weyl semimetals, arXiv:1512.01405.

P. V. Buividovich, M. Puhr, and S. N. Valgushev, Chiral magnetic conductivity in an interacting lattice model of parity-breaking Weyl semimetal, Phys. Rev. B **92**, 205122 (2015).

P. V. Buividovich, Spontaneous chiral symmetry breaking and the chiral magnetic effect for interacting Dirac fermions with chiral imbalance, Phys. Rev. D **90**, 125025 (2014).

P. V. Buividovich, Anomalous transport with overlap fermions, Nucl. Phys. **A925**, 218 (2014).

Chiral Magnetic Effect (CME) is the appearance of electric current in the direction of the external magnetic field in the presence of chiral chemical potential



Later the existance of the equilibrium static bulk CME was questioned.

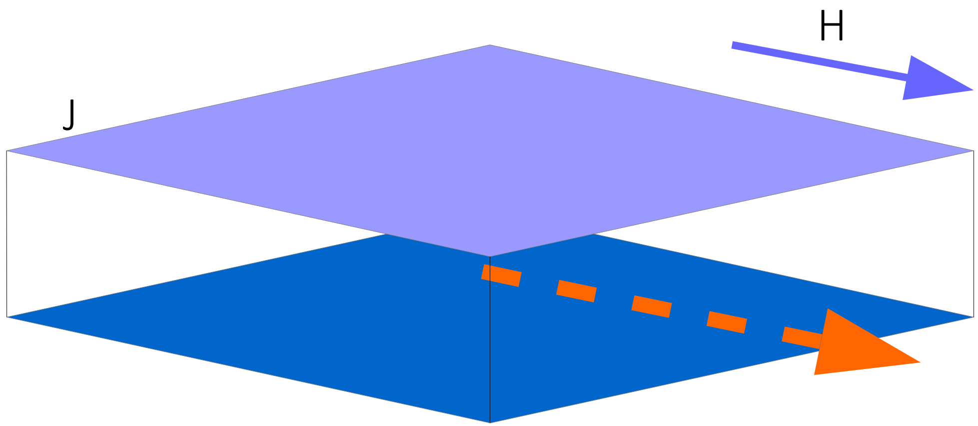
Weyl semimetals M. Vazifeh and M. Franz, Electromagnetic Response of Weyl Semimetals, Phys. Rev. Lett. **111**, 027201 (2013).

Analysis based on the attempt to apply Bloch theorem

.

N. Yamamoto, Generalized Bloch theorem and chiral transport phenomena, Phys. Rev. D **92**, 085011 (2015).

We start from the lattice model with massive fermions that describes lattice regularized quantum field theory or the insulators whose excitations are described by massive Dirac action (in solid state physics).



J = M /2pi^2 H   M = mu_5 ?

Chiral imbalance is described by the appearance of the chiral chemical potential

Green function (without external magnetic field) is:

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p}) \Big)^{-1}$$

Example : Wilson fermions

$$g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4} (1 - \cos p_a)$$

3+1 D Chiral Magnetic Effect

In lattice models
we obtain for the first time
$\mathcal{M}_4$ is responsible for the CME

$$j^{(1)k}(\mathbf{R}) = \frac{1}{4\pi^2}\epsilon^{ijkl}\mathcal{M}_l A_{ij}(\mathbf{R})$$

In continuous models this follows trivially from Feinman diagrams 4x4 Green function

$$\mathcal{M}_l = \int \mathrm{Tr}\, \nu_l \, d^4p$$

$$\nu_l = -\frac{i}{3!\,8\pi^2}\epsilon_{ijkl}\left[\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_i}\frac{\partial\mathcal{G}}{\partial p_j}\frac{\partial\mathcal{G}^{-1}}{\partial p_k}\right]$$

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1}$$

$$g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4}(1-\cos p_a)$$

$\mathcal{M}_4$ is responsible for the CME

$$\mathcal{M}_l = \int \mathrm{Tr}\, \nu_l \, d^4 p$$

$$\nu_l = -\frac{i}{3!\, 8\pi^2}\, \epsilon_{ijkl} \left[\mathcal{G} \frac{\partial \mathcal{G}^{-1}}{\partial p_i} \frac{\partial \mathcal{G}}{\partial p_j} \frac{\partial \mathcal{G}^{-1}}{\partial p_k} \right]$$

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4 \gamma^5 \mu_5 - i m(\mathbf{p}) \Big)^{-1} \qquad g_k(\mathbf{p}) = \sin p_k, \quad m(\mathbf{p}) = m^{(0)} + \sum_{a=1,2,3,4} (1 - \cos p_a)$$

(Wilson fermions at m_0 = 0 is the simplest model of Dirac semimetal)

m_0=0 mu_5≠0	⇔	m_0≠0 mu_5≠0	⇔	m_0≠0 mu_5=0

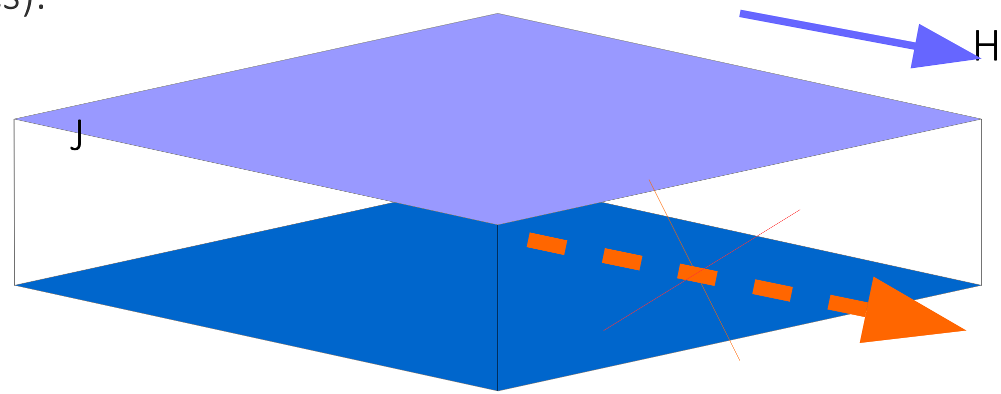
$\Uparrow$

$\mathcal{M}_4$=0

Poles of the Green function may appear for the nonzero mu_5 if

$$g_4^2(\mathbf{p}) + \Big(\mu_5 \pm \sqrt{g_1^2(\mathbf{p}) + g_2^2(\mathbf{p}) + g_3^2(\mathbf{p})} \Big)^2 + m^2(\mathbf{p}) = 0$$

We considered lattice models with both massive qnd massless fermions that describe lattice regularized quantum field theory or the insulators and Dirac semimetals whose excitations are described by massive/massless Dirac action (in solid state physics).



J = M_4 /2pi^2 H   M_4 = 0 as long as mu_5 is nonzero

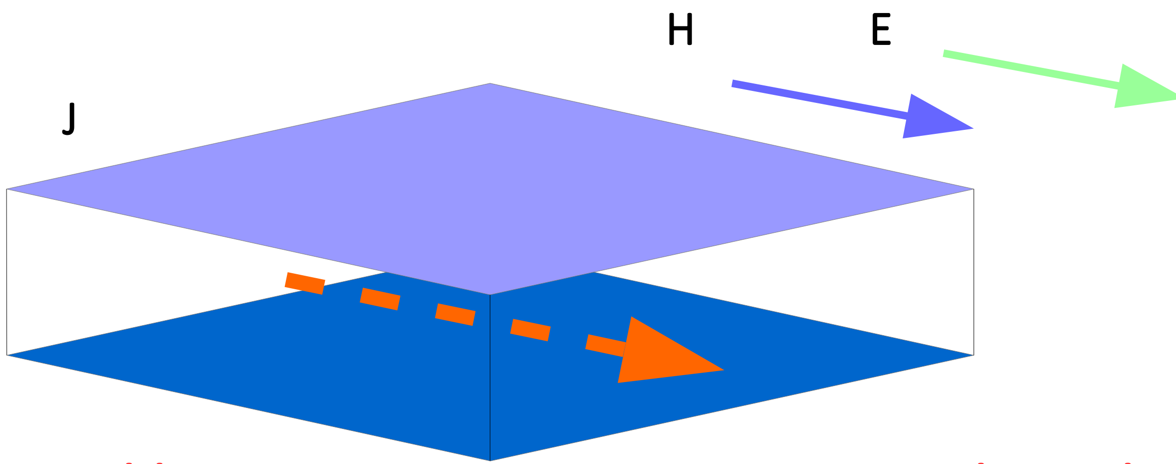
Chiral imbalance is described by the appearance of the chiral chemical potential

Green function (without external magnetic field) is:

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + i\gamma^4\gamma^5\mu_5 - im(\mathbf{p})\Big)^{-1}$$

There is no equilibrium CME

IN WHICH FORM THE CME MAY SURVIVE ?



1) nonequilibrium CME in Dirac semimetals in the presence of real or emergent magnetic field (say, due to the dislocations)

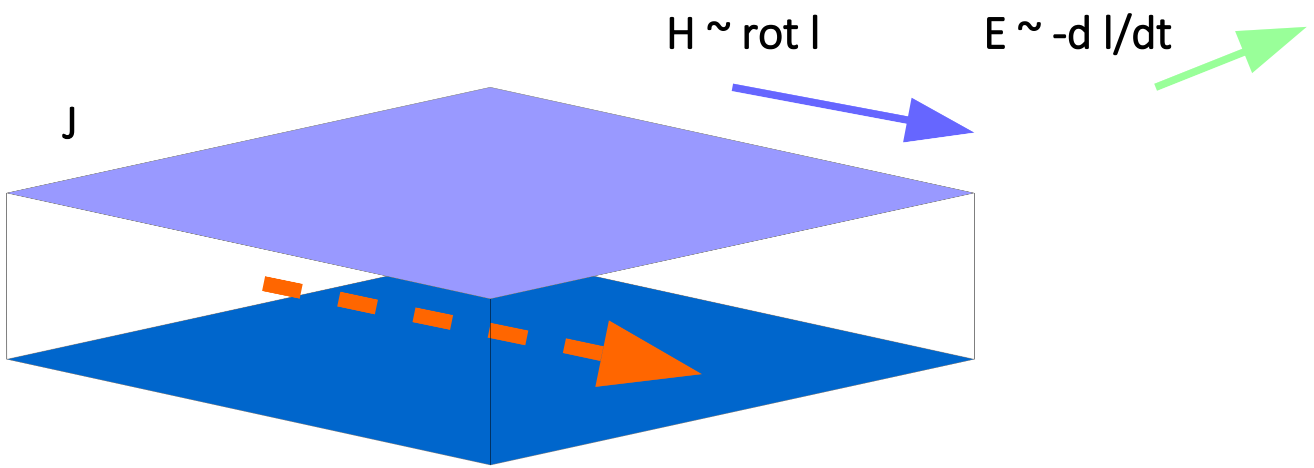
the chiral anomaly produces chiral Imbalance

this production requires energy taken from the job performed by the electic field.

This assumes existence of electric current j

JE = energy created while pumping the pairs from vacuum =>
J = ?

IN WHICH FORM THE CME MAY SURVIVE ?

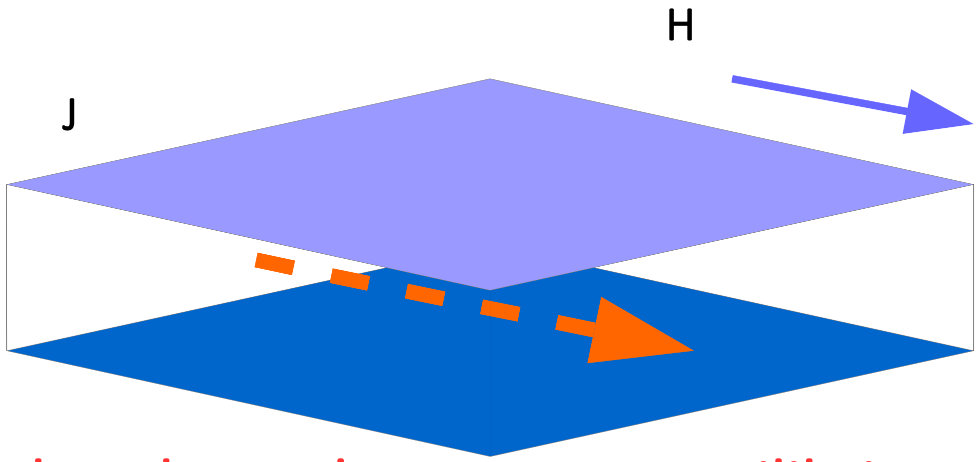


2) CME in He3-A, where mu_5 ~ l (v_n-v_s), L ~ mu_5 A H

The applied technique for the calculation of the CME current does not work here because :

- the problem is not equilibrium
- the gauge field is emergent rather than real

IN WHICH FORM THE CME MAY SURVIVE ?



3) Quark – gluon plasma : nonequilibrium CME contributions to the kinetic equations in the presence of the chiral imbalance?

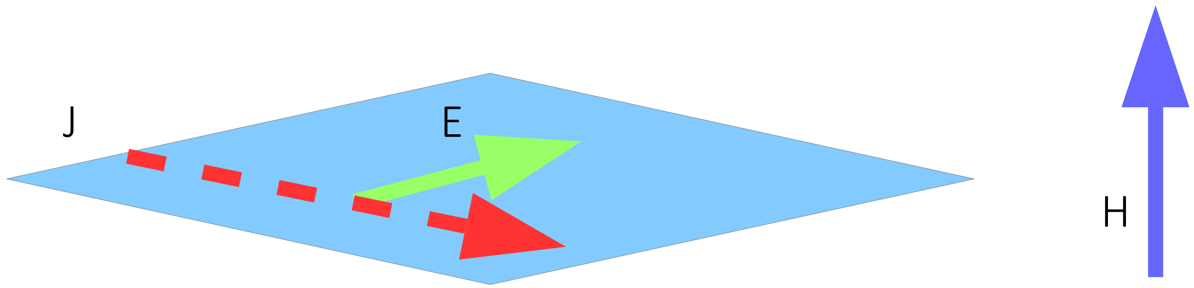
Chiral imbalance that is described by chiral density rather than the chiral chemical potential ?

Applications

2. AQHE in 2+1 D, 3+1 D topological insulators, and in 3+1 D Weyl semimetals

M.A.Zubkov, « Wigner transformation, momentum space topology, and anomalous transport » arXiv:1603.03665, Annals Phys. 373 (2016) 298-324

Hall effect is the appearance of electric current in the direction orthogonal to the external magnetic field and external electric field.

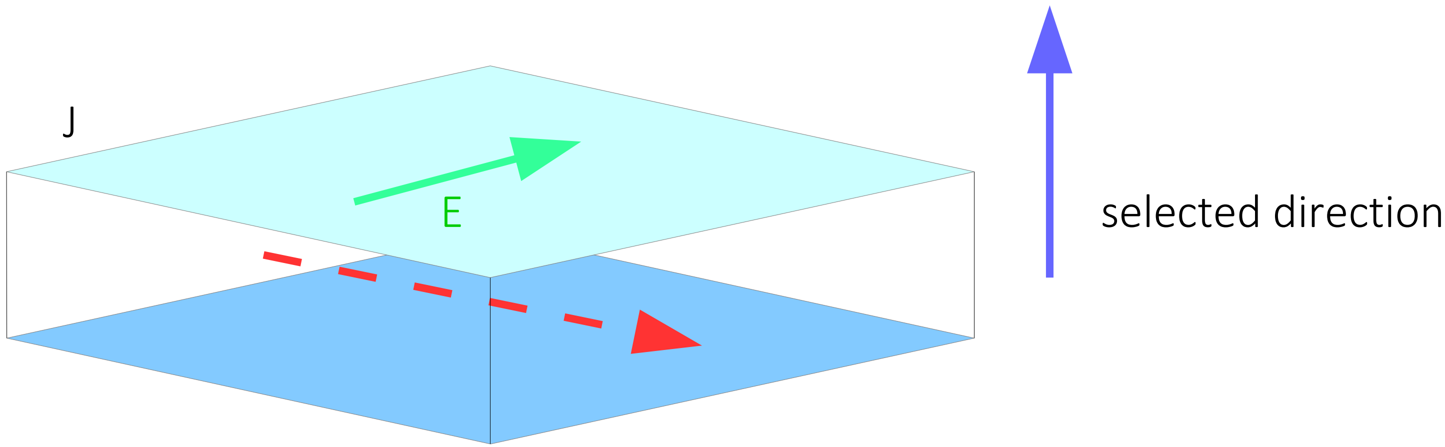


Qauntum Hall effect is the quantized Hall effect

J = N/2pi E

Anomalous Quantum Hall (AQHE) effect is the appearance of quantized current orthogonal to electric field without any external magnetic field (experimentally discovered in 2D materials).

AQHE Hall effect is the appearance of electric current in the direction orthogonal to the external electric field.



J = M /4pi^2 E

Weyl semimetals

M = [distance between the Weyl points in momentum space]

Topological insulators

M = const/a = integer x [vector of inverse lattice]

Response of the electric current to the gauge field

$$j^k(\mathbf{R}) = j^{(0)k}(\mathbf{R}) + j^{(1)k}(\mathbf{R}) + \ldots$$

$$j^{(0)k}(\mathbf{R}) = \int \frac{d^D\mathbf{p}}{(2\pi)^D} \operatorname{Tr} \tilde{G}^{(0)}(\mathbf{R},\mathbf{p}) \frac{\partial \left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]^{-1}}{\partial p_k}$$

2+1 D

$$j^{(1)k}(\mathbf{R}) = \frac{1}{4\pi}\epsilon^{ijk}\mathcal{M} A_{ij}(\mathbf{R}), \quad \mathcal{M} = \int \operatorname{Tr} \nu \, d^3p$$

$$\nu = -\frac{i}{3!\,4\pi^2}\,\epsilon_{ijk}\left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\frac{\partial\left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]^{-1}}{\partial p_i}\frac{\partial\left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]}{\partial p_j}\frac{\partial\left[\tilde{G}^{(0)}(\mathbf{R},\mathbf{p})\right]^{-1}}{\partial p_k}\right]$$

2+1 D Anomalous Quantum Hall effect clean 2D top. Insulator

We reproduce

$\mathbf{E} = (E_1, E_2)$ as $A_{3k} = -iE_k$

$$j^k_{Hall} = \frac{1}{2\pi} \tilde{\mathcal{N}}_3 \, \epsilon^{ki} E_i$$

$$\tilde{\mathcal{N}}_3 = -\frac{1}{24\pi^2} \mathrm{Tr} \int \mathcal{G}^{-1} d\mathcal{G} \wedge d\mathcal{G}^{-1} \wedge d\mathcal{G}$$

G.E.Volovik, JETP 67 (1988), 1804 — 1811

G.E.Volovik

(Momentum space topology in condensed matter and beyond)

2+1 D Anomalous Quantum Hall effect clean 2D top. Insulator

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G.E.Volovik, JETP 67 (1988), 1804 — 1811

In the particular case of the non — interacting system it is ' d to

$$\mathcal{G}^{-1} = i\omega - \hat{H}$$

$$\tilde{\mathcal{N}}_3 = \frac{\epsilon^{ij}}{4\pi}\sum_{k:\mathcal{E}_k<0}\int d^2p\,\mathcal{F}_{ij}$$

$$\mathcal{F}_{ij} = \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i \qquad \mathcal{A}_j = i\langle k,\vec{p}|\partial_j|k,\vec{p}\rangle$$

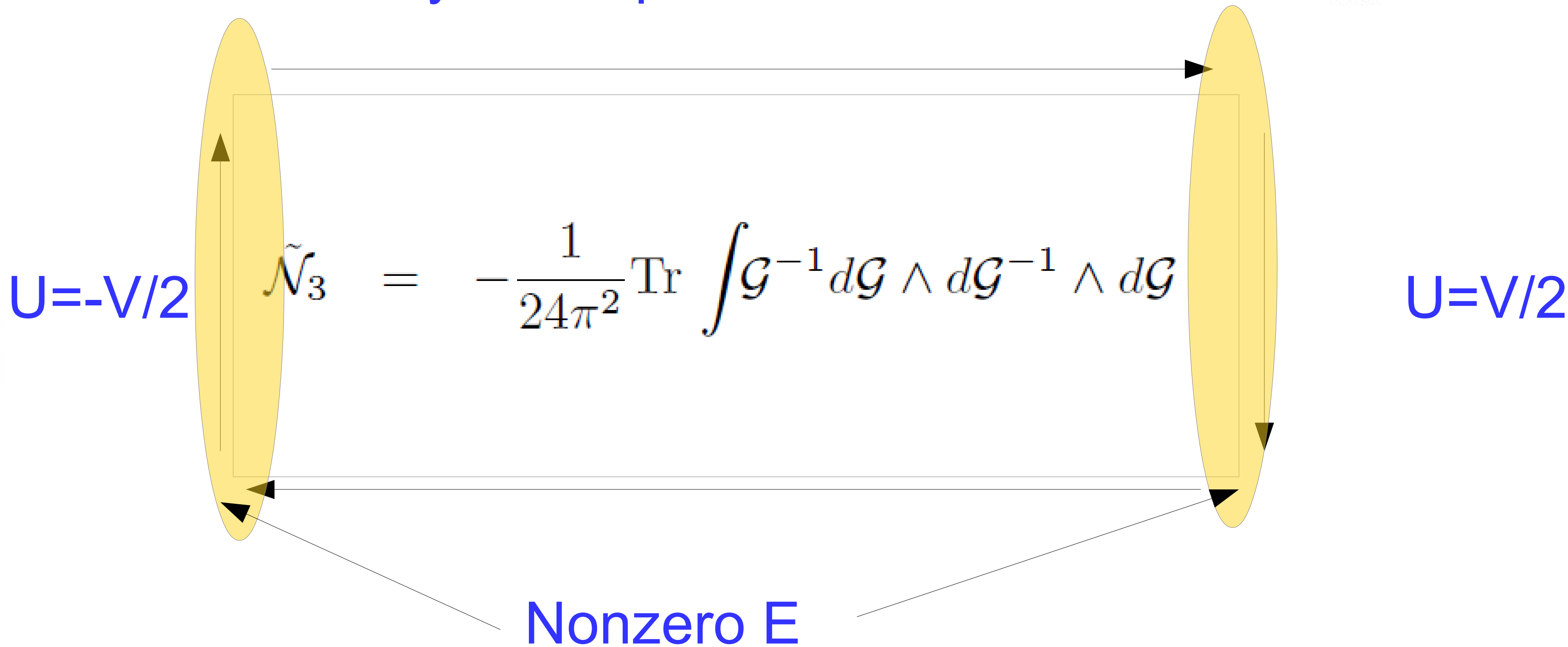
Berry curvature

D. J. Thouless, M. Kohmoto, M. P. Nightingale, M. den Nijs, Phys. Rev. Lett. 49, 405 (1982)

2+1 D Anomalous Quantum Hall effect

Bulk — boundary correspondence

$$j^k_{Hall} = \frac{1}{2\pi}\tilde{\mathcal{N}}_3\,\epsilon^{ki}E_i$$



$$\tilde{\mathcal{N}}_3 = -\frac{1}{24\pi^2}\mathrm{Tr}\int \mathcal{G}^{-1}d\mathcal{G}\wedge d\mathcal{G}^{-1}\wedge d\mathcal{G}$$

In the presence of disorder the total current is given by
N_3/2pi x V = J_left — J_right
and is carried by the boundary gapless fermions
(we use relativistic units)

3+1 D Anomalous Quantum Hall effect 3D top. insulator

We obtain for the first time

$$j^k_{Hall} = \frac{1}{4\pi^2}\,\mathcal{M}'_l\,\epsilon^{jkl}\,E_j$$

$$\mathcal{M}'_l = \frac{1}{3!\,4\pi^2}\,\epsilon_{ijkl}\int d^4p\,\mathrm{Tr}\Big[\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_i}\frac{\partial\mathcal{G}}{\partial p_j}\frac{\partial\mathcal{G}^{-1}}{\partial p_k}\Big]$$

In the particular case of the non — interacting system it is reduced to

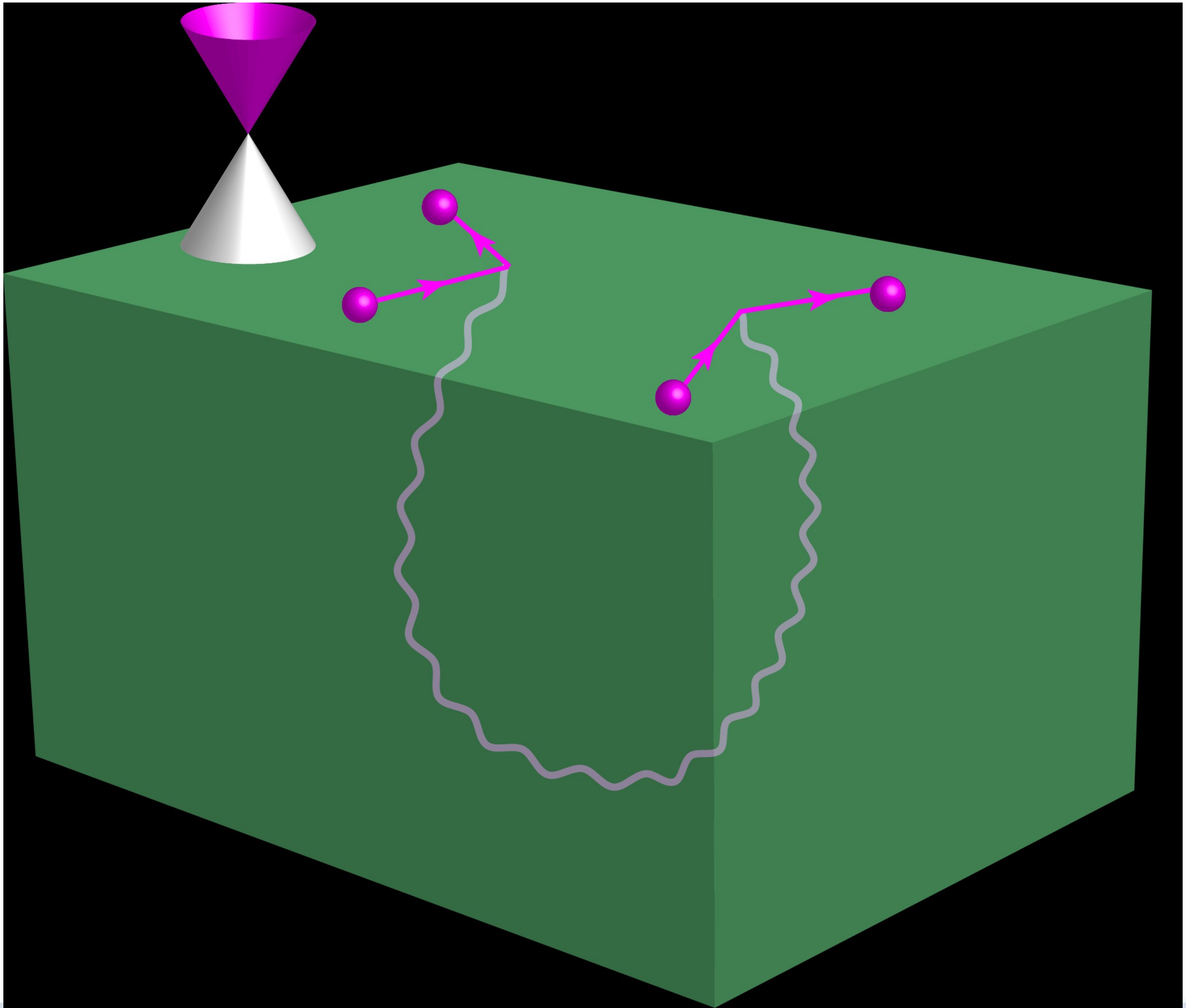
$$\mathcal{G}^{-1} = i\omega - \hat{H}$$

$$\mathcal{M}'_l = \frac{\epsilon^{ijl}}{4\pi}\sum_{\mathrm{occupied}}\int d^3p\,\mathcal{F}_{ij}$$

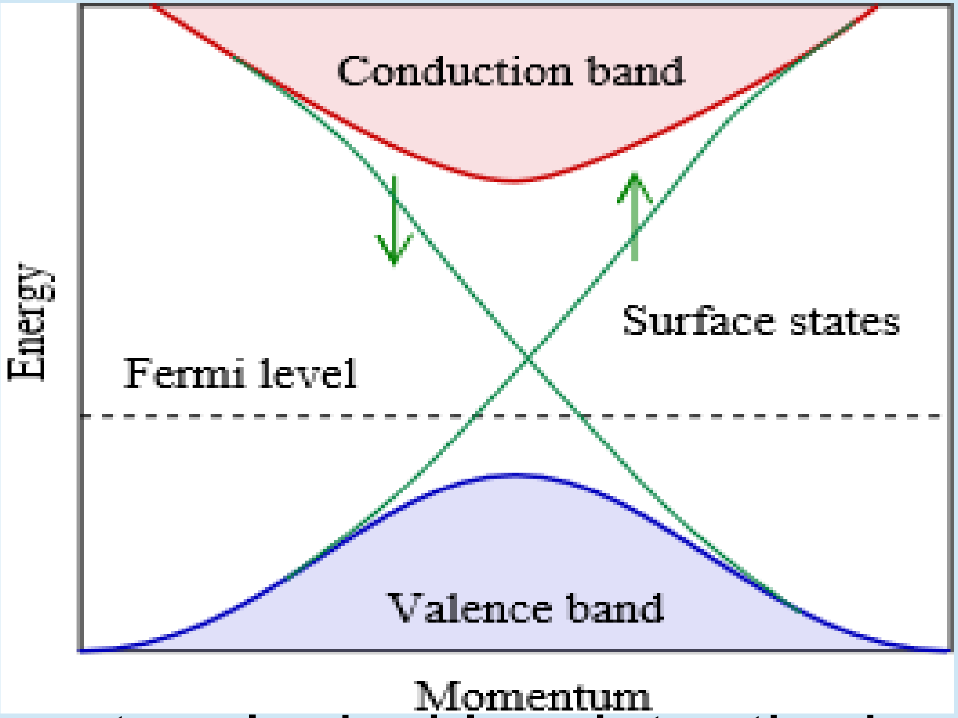
Berry curvature

$$\mathcal{F}_{ij} = \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i \qquad \mathcal{A}_j = i\langle k,\vec{p}|\partial_j|k,\vec{p}\rangle$$

An ordinary
topological insulator



A **topological insulator** is a material with non-trivial topological order that behaves as an insulator in its interior but whose surface contains conducting states meaning that electrons can only move along the surface of the material.



For the ordinary topological Insulator the boundary gapless states correspond to the Fermi points (the points in momentum space where particle energy vanishes)

3+1 D Anomalous Quantum Hall effect 3D top. insulator

We obtain for the first time

$$j^k_{Hall} = \frac{1}{4\pi^2}\mathcal{M}'_l\,\epsilon^{jkl}E_j$$

$$\mathcal{M}'_l = \frac{1}{3!\,4\pi^2}\epsilon_{ijkl}\int d^4p\,\mathrm{Tr}\Big[\mathcal{G}\frac{\partial\mathcal{G}^{-1}}{\partial p_i}\frac{\partial\mathcal{G}}{\partial p_j}\frac{\partial\mathcal{G}^{-1}}{\partial p_k}\Big]$$

2x2 Green function

$$\mathcal{G}^{-1}(\mathbf{p}) = i\sigma^3\Big(\sum_k \sigma^k g_k(\mathbf{p}) - ig_4(\mathbf{p})\Big)$$

Sum over points, where g_k=0 (k=1,2,3)

$$\hat{g}_k = \frac{g_k}{g}$$

$$\mathcal{M}'_j = -\frac{1}{2}\sum_l\int_{y_l(s)}\mathrm{sign}(g_4(y_l))\,\mathrm{Res}\,(y_l)dp_j$$

$$g = \sqrt{\textstyle\sum_{k=1,2,3,4} g_k^2}$$

$$\mathrm{Res}\,(y_l) = \frac{1}{8\pi}\epsilon^{ijk}\int_{\Sigma(y_l)}\hat{g}_i d\hat{g}_j \wedge d\hat{g}_k$$

3+1 D Anomalous Quantum Hall effect 3D top. insulator

Example $\mathcal{G}^{-1} = i\omega - \hat{H}$ $j^k_{Hall} = \frac{1}{4\pi^2} \mathcal{M}'_l \, \epsilon^{jkl} E_j$

$$H = \sin p_1 \, \sigma^2 - \sin p_2 \, \sigma^1 - (m^{(0)} - \gamma \cos p_3 + \sum_{i=1,2} (1 - \cos p_i)) \, \sigma^3$$

$$\gamma < 1, \text{ and } m^{(0)} \in (-2+\gamma, -\gamma)$$

$$\mathcal{M}'_3 = \frac{2\pi}{2} - \frac{2\pi}{2}(-1) - \frac{2\pi}{2}(-1) - \frac{2\pi}{2} = 2\pi$$

$$j^k_{Hall} = \frac{1}{2\pi \, a} \, \epsilon^{jk3} E_j$$

3+1 D Anomalous Quantum Hall effect 3D top. insulator

Example $\mathcal{G}^{-1} = i\omega - \hat{H}$ $\quad j^k_{Hall} = \frac{1}{4\pi^2}\mathcal{M}'_l\,\epsilon^{jkl}E_j$

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + \gamma^5 g_5(\mathbf{p}) + \gamma^3\gamma^5 b(\mathbf{p})\Big)^{-1} \qquad g_3^{(0)} > 1,\ m^{(0)} > g_3^{(0)} - 1$$

$$g_1(\mathbf{p}) = -\sin p_2, \quad g_2(\mathbf{p}) = \sin p_1, \quad g_3(\mathbf{p}) = g_3^{(0)} + \sin p_3$$

$$g_4(\mathbf{p}) = \omega, \quad g_5(\mathbf{p}) = m^{(0)} + \sum_{a=1,2}(1-\cos p_a), \quad b = const$$

$$\sqrt{(g_3^{(0)}+1)^2 + (m^{(0)})^2} < b$$

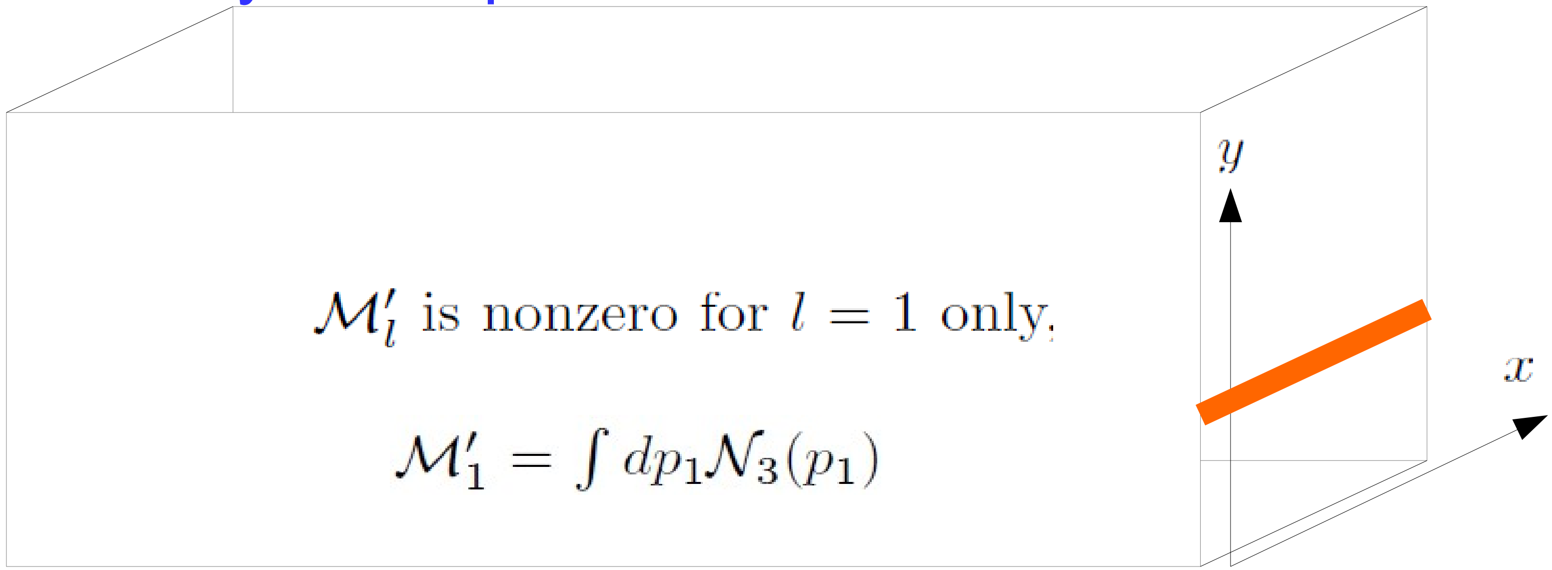
$$\sqrt{(g_3^{(0)}-1)^2 + (m^{(0)}+2)^2} > b$$

$$\mathcal{M}'_3 \;=\; \frac{2\pi}{2} - \frac{2\pi}{2}(-1) - \frac{2\pi}{2}(-1) - \frac{2\pi}{2} = 2\pi \qquad j^k_{Hall} = \frac{1}{2\pi\, a}\epsilon^{jk3}E_j$$

3+1 D AQH, top. insulator

$$j^k_{Hall} = \frac{1}{4\pi^2}\mathcal{M}'_l\,\epsilon^{jkl}E_j$$

Bulk — boundary correspondence



$\mathcal{M}'_l$ is nonzero for $l = 1$ only.

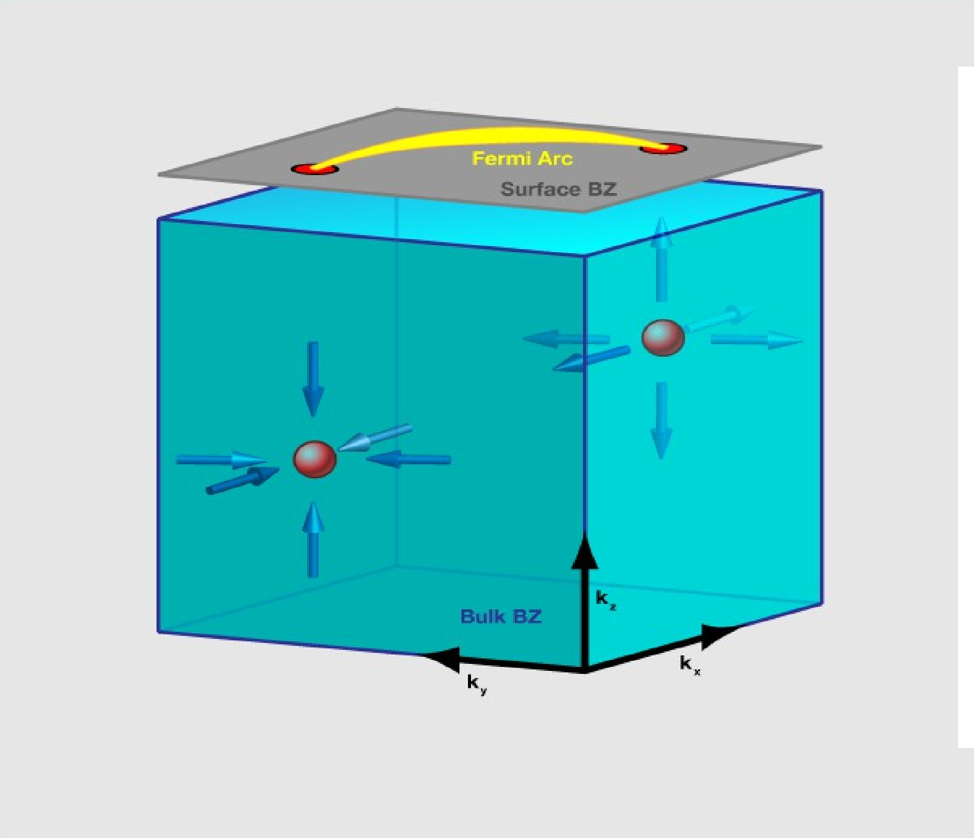
$$\mathcal{M}'_1 = \int dp_1 \mathcal{N}_3(p_1)$$

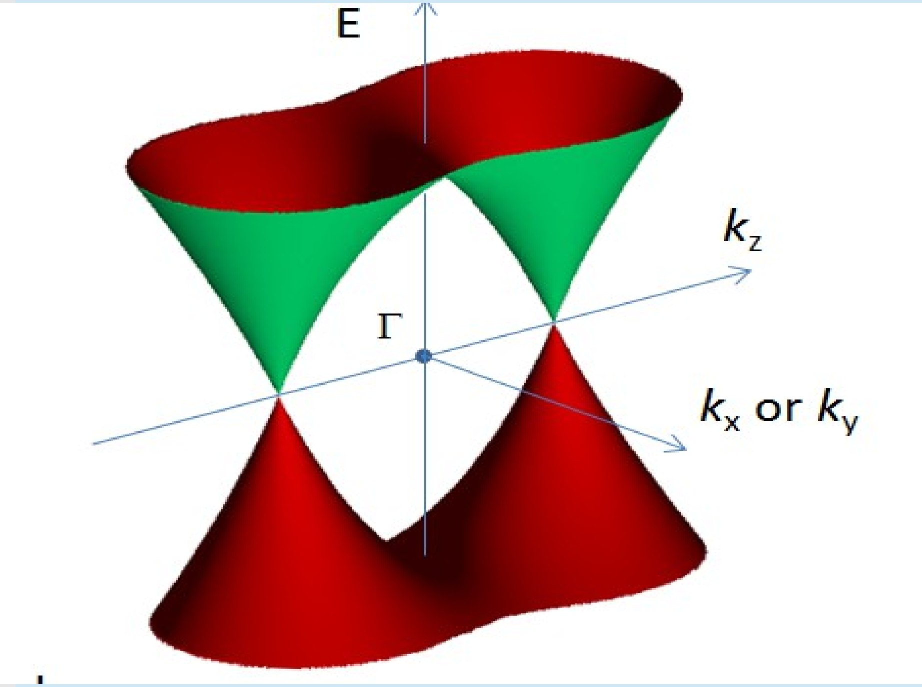
$$\mathcal{N}_3(p_1) \quad = \quad \frac{1}{3!\,4\pi^2}\,\epsilon_{ijkl}\int dp_2dp_3dp_4\,\mathrm{Tr}\Big[\tilde{G}^{(0)}\frac{\partial(\tilde{G}^{(0)})^{-1}}{\partial p_i}\frac{\partial\tilde{G}^{(0)}}{\partial p_j}\frac{\partial(\tilde{G}^{(0)})^{-1}}{\partial p_k}\Big]$$

Index theorem: at each value of p_1 the jump of N_3 is equal to the number of gapless chiral boundary modes.

We have N_3 Fermi lines on the xy and xz boundaries and N_3 Fermi points on the yz boundary

A **Weyl semimetal** is a solid state crystal whose low energy excitations are Weyl fermions. A Weyl semimetal enables the first-ever realization of Weyl fermions. It is a topologically nontrivial phase of matter that broadens the topological classification beyond topological insulators





3+1 D Anomalous Quantum Hall effect for Weyl semimetal

Example $\mathcal{G}^{-1} = i\omega - \hat{H}$ $j^k_{Hall} = \frac{1}{4\pi^2}\mathcal{M}'_l\,\epsilon^{jkl}E_j$

$$\mathcal{G}(\mathbf{p}) = \Big(\sum_k \gamma^k g_k(\mathbf{p}) + \gamma^5 g_5(\mathbf{p}) + \gamma^3\gamma^5 b(\mathbf{p})\Big)^{-1}$$

$$g_1(\mathbf{p}) = -\sin p_2, \quad g_2(\mathbf{p}) = \sin p_1, \quad g_3(\mathbf{p}) = g_3^{(0)} + \sin p_3$$

$$g_4(\mathbf{p}) = \omega, \quad g_5(\mathbf{p}) = m^{(0)} + \sum_{a=1,2}(1-\cos p_a), \quad b = const$$

$$g_3^{(0)} > \sqrt{b^2 - (m^{(0)})^2} > g_3^{(0)} - 1 > 0 \qquad \sqrt{(g_3^{(0)} + \sin\beta_\pm)^2 + (m^{(0)})^2} = b$$

Two Fermi points $\mathbf{K}_\pm = (0,0,\beta_\pm,0)$

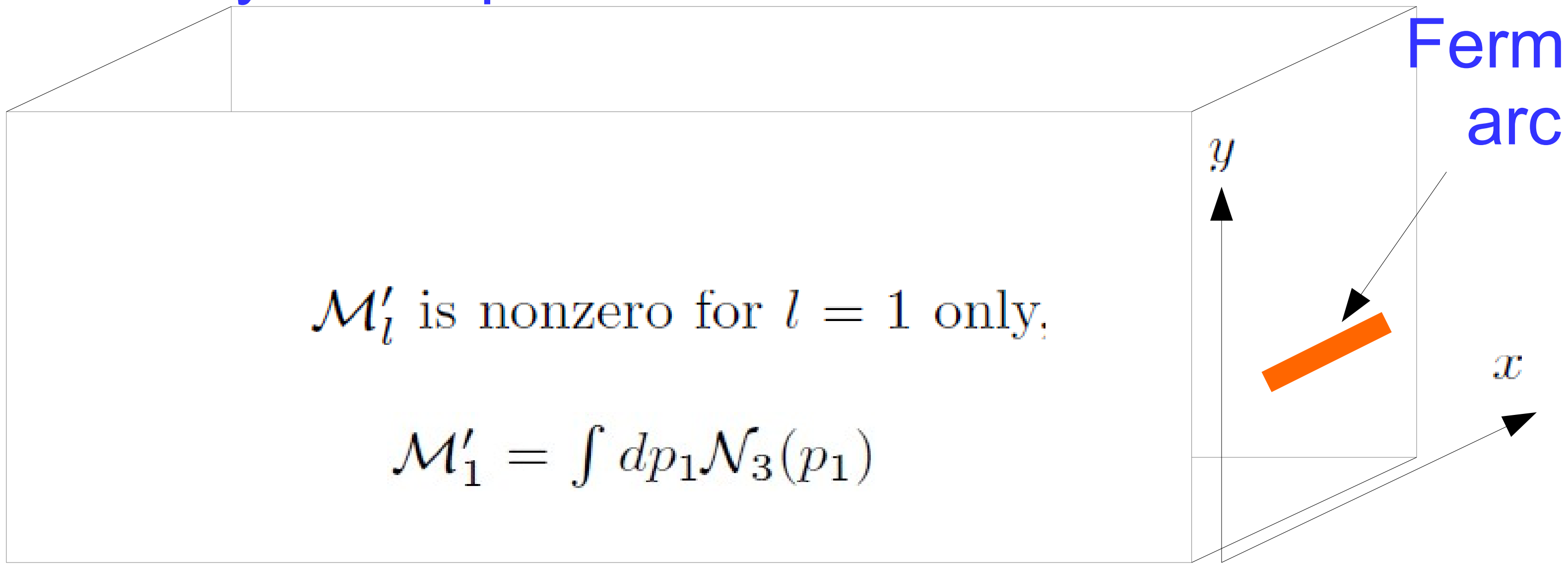
$$j^k_{Hall} = \frac{\beta_+ - \beta_-}{4\pi^2}\,\epsilon^{jk3}E_j$$

the same expression may be obtained using effective continous QFT

3+1 D AQH, Weyl semimetal

Bulk — boundary correspondence

$$j^k_{Hall} = \frac{1}{4\pi^2} \mathcal{M}'_l \, \epsilon^{jkl} E_j$$



$$\mathcal{N}_3(p_1) \;=\; \frac{1}{3!\,4\pi^2}\,\epsilon_{ijkl} \int dp_2 dp_3 dp_4 \,\mathrm{Tr}\Big[\tilde{G}^{(0)} \frac{\partial(\tilde{G}^{(0)})^{-1}}{\partial p_i} \frac{\partial \tilde{G}^{(0)}}{\partial p_j} \frac{\partial(\tilde{G}^{(0)})^{-1}}{\partial p_k}\Big]$$

Index theorem: at each value of p_1 the jump of N_3 is equal to the number of gapless chiral boundary modes.

We have N_3 Fermi arcs on the xy and xz boundaries that connect the bulk Fermi points, and N_3 Fermi points on the yz boundary

Weyl semimetal TaAs

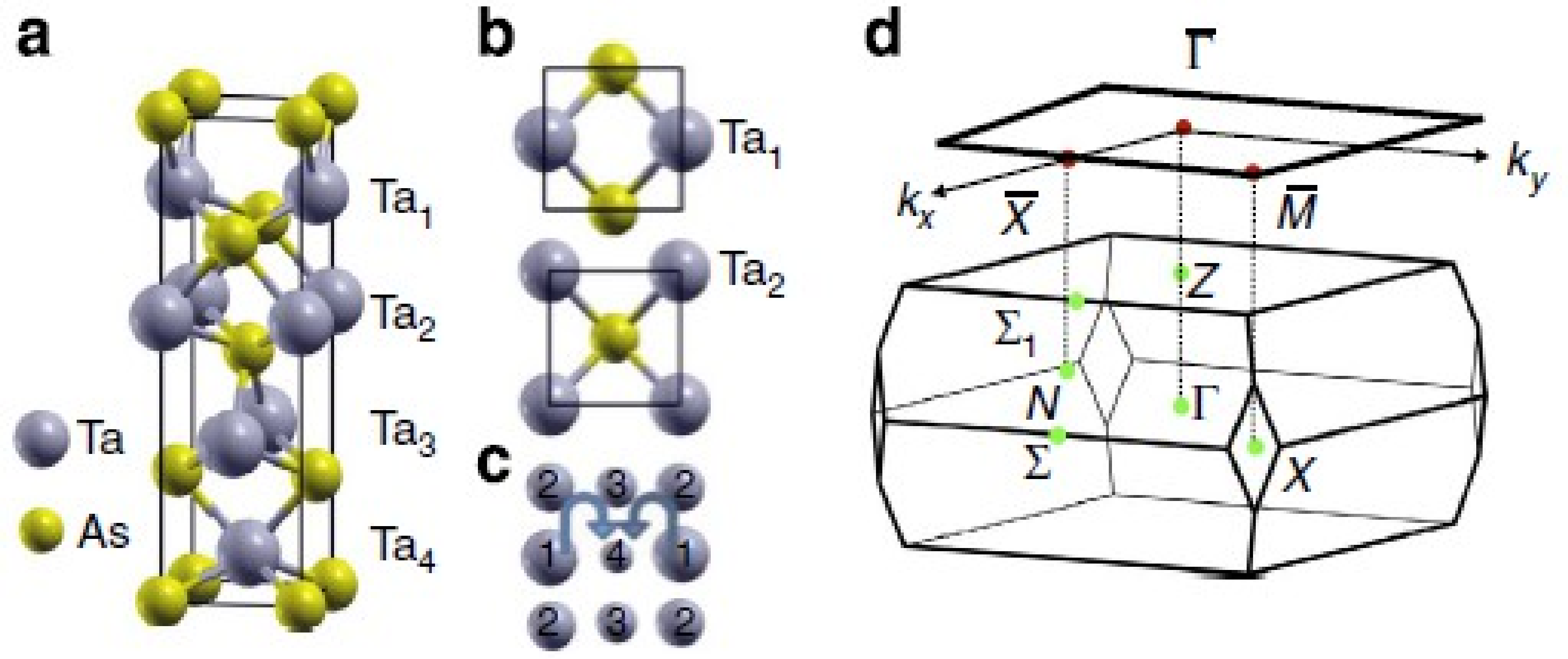
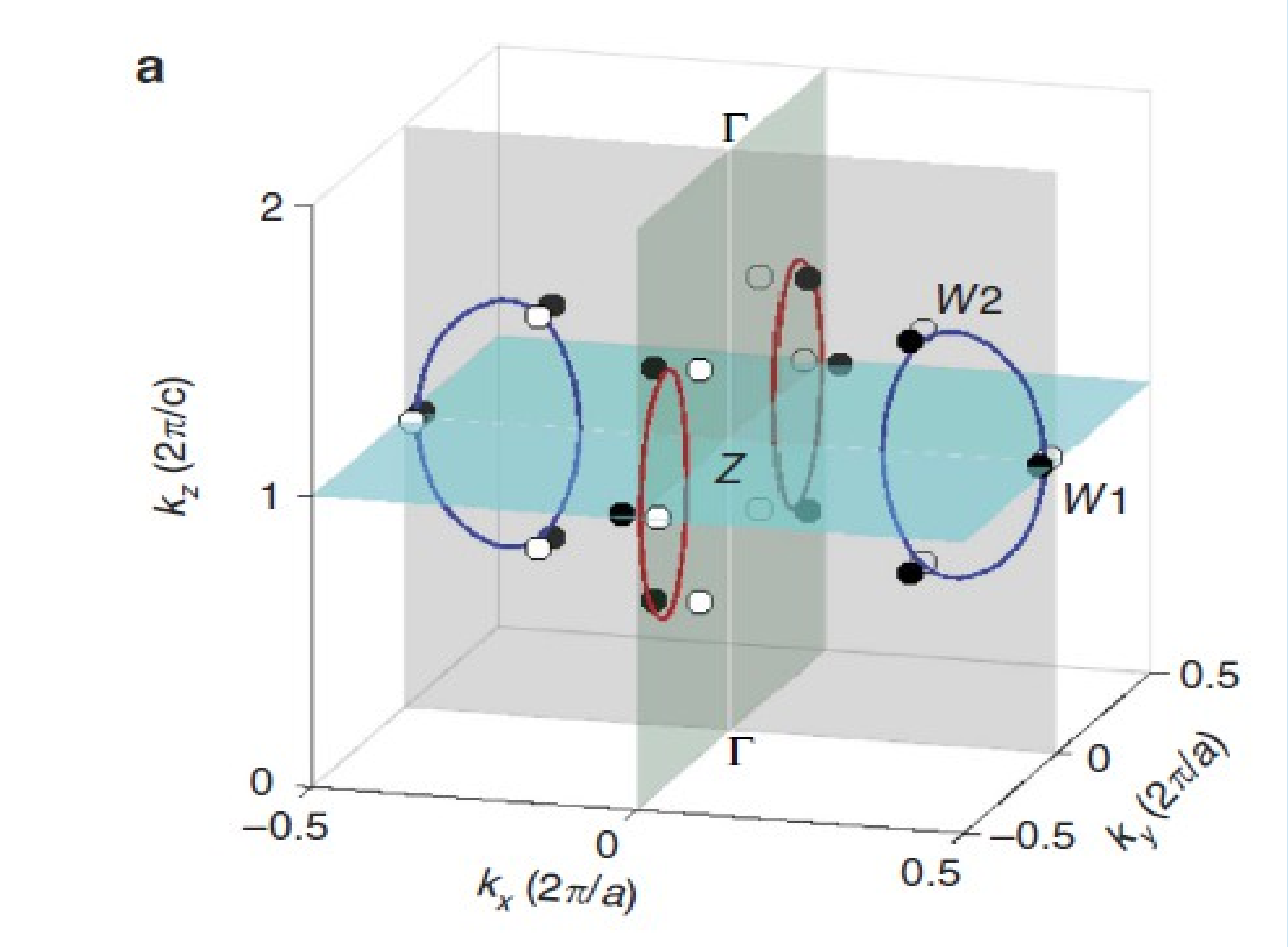


Figure 1 | Crystal structure and Brillouin zone of the Weyl semimetal

Weyl semimetal TaAs

12x2 Weyl points of two types 8 x W1 + 16 x W2

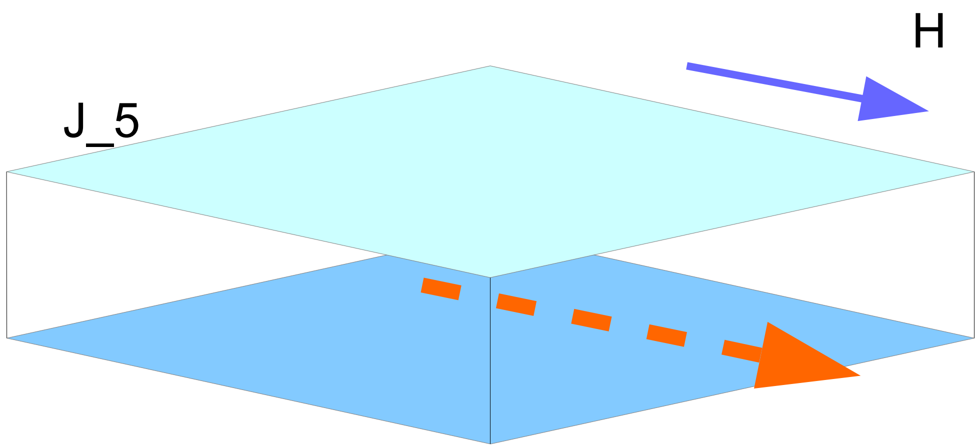


Applications

3. Chiral Separation Effect for massless or nearly massless systems

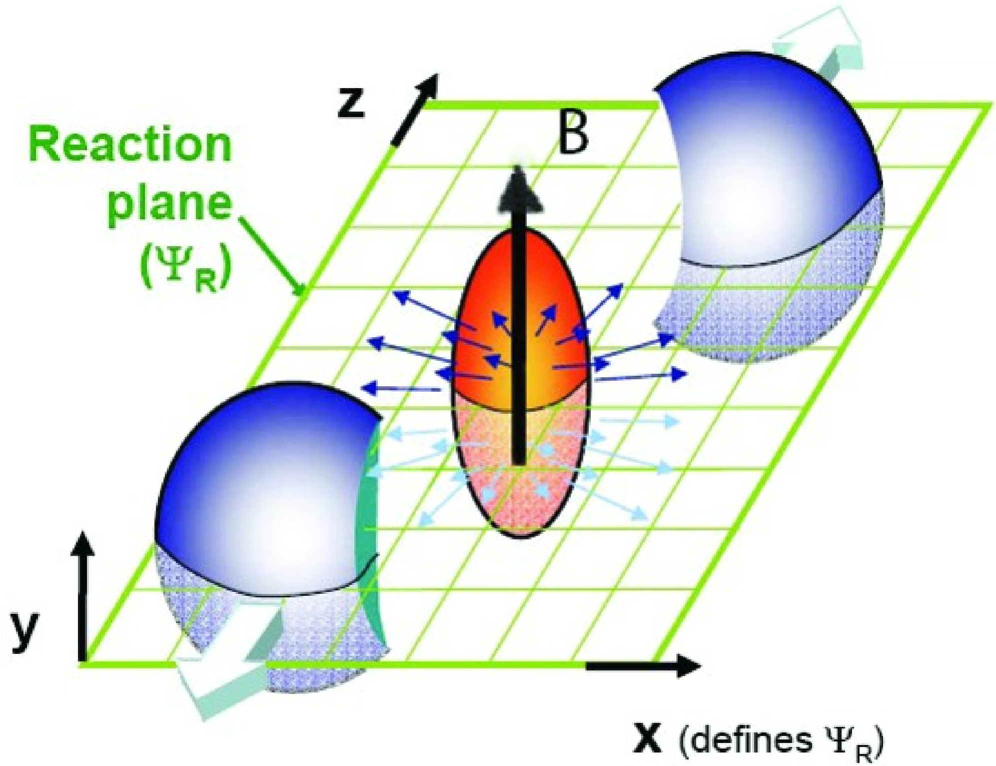
Z.V.Khaidukov, M.A.Zubkov, "Chiral Separation effect in lattice regularization" Phys. Rev. D 95 (2017), 074502

Chiral Separation Effect (CME) is the appearance of axial current in the direction of the external magnetic field in the presence of chemical potential



J_5 = M /2pi^2 H   M = mu

A. Metlitski and Ariel R. Zhitnitsky,Phys. Rev. D 72 (2005), 045011



z
B
Reaction
plane
(Ψ_R)
y
x (defines Ψ_R)

CSE is the appearance of axial current along the external magnetic field

$$\bar{\mathrm{j}}^5 = -\frac{\mu}{2\pi^2}\overline{\mathrm{B}}$$

μ - chemical potential, $\mathrm{j}^{5,\mathrm{k}} = \langle\bar{\psi}\gamma^{\mathrm{k}}\gamma^5\rangle$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Anomalous Axion Interactions and Topological Currents in Dense Matter Max A. Metlitski and Ariel R. Zhitnitsky, Phys. Rev. D 72, 045011

Anomalous Axion Interactions and Topological Currents in Dense Matter Max A. Metlitski and Ariel R. Zhitnitsky, Phys. Rev. D 72, 045011

The current has been calculated via the direct construction of spectrum in the presence of external field. The Hamiltonian is

$$H = -i(\partial_i + ieA_i)\gamma^0\gamma^i + m\gamma^0$$

In the presence of exact chiral symmetry

In Euclidian space - time $j^{5,k}(R) = -\frac{\delta}{\delta A^5_k} Z[A^5]$, где
$Z = \int D\bar{\psi} D\psi \exp\Big(- \int_{\mathcal{M}} \frac{d^D p}{|\mathcal{M}|} \bar{\psi}^T(p) \mathcal{G}^{-1}(p - A^5\gamma^5)\psi(p)\Big)$

$$j^{5,k}(R) \quad = \quad \int_{\mathcal{M}} \frac{d^D p}{(2\pi)^D} \, \mathrm{Tr}\, \gamma^5 \tilde{G}(R,p) \frac{\partial}{\partial p_k} \Big[\tilde{G}^{(0)}(R,p)\Big]^{-1}$$

where $\mathcal{G}(p) = -i\Big(\sum_k \gamma^k g_k(p) - im(p)\Big)^{-1}$ и

$$\tilde{G}^{(0)}(R,p) = \mathcal{G}(p - \gamma^5 A^5(R))$$

$$j^{5k}(R) = \int_{\mathcal{M}} \frac{d^D p}{(2\pi)^D} \mathrm{Tr}\, \gamma^5 \tilde{G}(R,p) \frac{\partial}{\partial p_k} \left[\tilde{G}^{(0)}(R,p)\right]^{-1}$$

Let us use this expression as the definition of the chiral current in the lattice theory in the absence of exact chiral symmetry. In the naive continuum limit this definition gives $-i\bar{\psi}\gamma^k\gamma^5\psi$ (Euclid) $\bar{\psi}\gamma^k\gamma^5\psi$ (Minkowski)

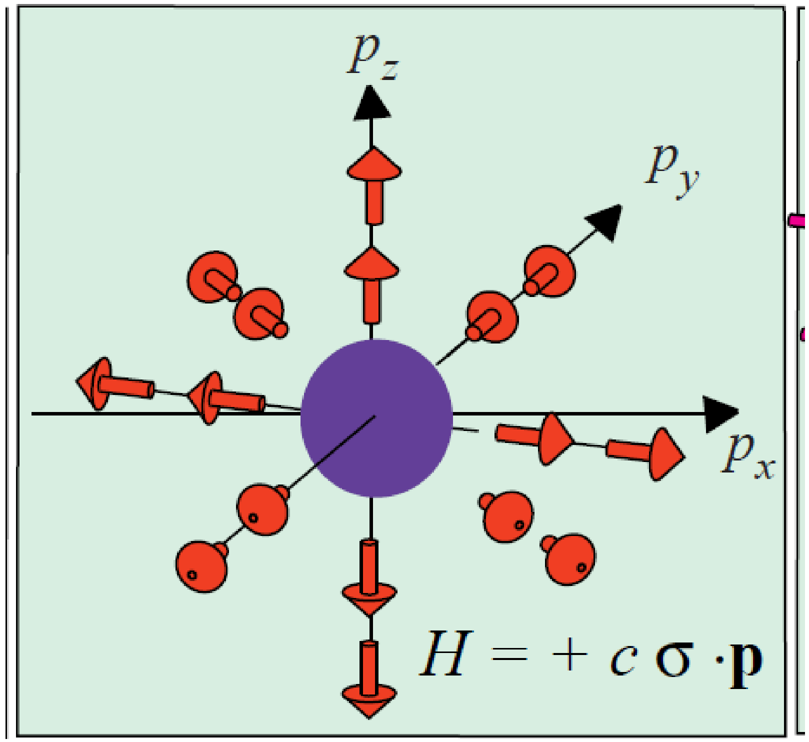
The term linear in chemical potential

$$j^{5k} = \frac{\mathcal{N}\,\epsilon^{ijk}}{4\pi^2} F_{ij}\mu$$

Here

$$\mathcal{N} = \frac{1}{12}\int_{\Sigma} \frac{1}{(2\pi)^2} Tr\gamma^5 \mathcal{G}(\omega,\mathbf{p}) d\mathcal{G}^{-1}(\omega,\mathbf{p}) \wedge d\mathcal{G}(\omega,\mathbf{p}) \wedge d\mathcal{G}^{-1}(\omega,\mathbf{p})$$

The surface surrounds the Fermi point in momentum space



Weyl point - hedgehog in p-space

For the case of Wilson fermions

$$\mathrm{G}^{-1}(\mathrm{p}) = \sum_{\mathrm{i}=1..4} \sin(\mathrm{p_i})\gamma^{\mathrm{i}} + (\sum_{\mathrm{i}=1..4} 2\sin^2(\mathrm{p_i}/2) + \mathrm{m}^{(0)})\mathbf{i}$$

Poles may appear at $\omega = 0, \pi$. For $m^{(0)} \to 0$ there is the pole only at $\omega = 0$

$$\mathcal{N} = 1$$

Conclusions

1. The formalism of Wigner transformation has been applied to the Green functions defined on the lattice.

2. Using the derivative expansion we express the non – dissipative currents through the topological invariant in momentum space

* AQHE in the 2D topological insulators (Anomalous Quantum Hall Effect)

$$j^k_{Hall} = \frac{1}{2\pi} \tilde{\mathcal{N}}_3 \, \epsilon^{ki} E_i$$

* AQHE in the 3D topological insulators

$$j^k_{Hall} = \frac{1}{4\pi^2} \mathcal{M}'_l \, \epsilon^{jkl} E_j$$

* AQHE in Weyl semimetals.

* CME in Dirac semimetals and in the QFT with massless fermions (Chiral Magnetic Effect)

$$j^{(1)k}(\mathbf{R}) = \frac{1}{4\pi^2} \epsilon^{ijkl} \mathcal{M}_l A_{ij}(\mathbf{R})$$

* CSE in Dirac semimetals and in the QFT With massless fermions (the Chiral Separation Effect)

$$j^{5k} = \frac{\mathcal{N} \, \epsilon^{ijk}}{4\pi^2} F_{ij} \mu$$