

Tangle blocks and polynomial calculus

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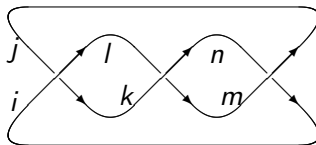
30 May, Quarks-2018

Plan of the talk

- Approaches to \mathcal{R} -matrix calculus of knot polynomials
 - Universal \mathcal{R} -matrix
 - Braid presentation of knot
 - Two-bridge knots
 - Arborescent knots
- Tangles and tangle blocks
- Simple tangle blocks

\mathcal{R} -matrix approach

One of the most useful approaches to calculating knot polynomials is to use \mathcal{R} -matrices.



The answer for polynomials is given by a product of such matrices:

$$H = \mathcal{R}_{ij}^{kl} \mathcal{R}_{kl}^{mn} \mathcal{R}_{mn}^{ij}$$

All indices run over all vectors in the representations.

Universal \mathcal{R} -matrix

These matrices are solutions of Yang-Baxter equation and there exist an exact formula for this matrix:

$$\mathcal{R} = \mathcal{P} q^{\sum_{\alpha} h_{\alpha} \otimes h_{\alpha}} \prod_{\beta} \exp_q \left((q_{\beta} - q_{\beta}^{-1}) E_{\beta} \otimes F_{\beta} \right)$$

It is awfully complex structure which is hard to calculate and use. For the simplest case of fundamental representation of $SU_q(2)$ it is equal to

$$\mathcal{R} = \begin{pmatrix} q & & & \\ & q - q^{-1} & 1 & \\ & 1 & & \\ & & & q \end{pmatrix}$$

Space of intertwining operators

\mathcal{R} -matrix is much more understandable if one looks at it in the space of intertwining operators, i.e. in the space of irreducible representations in the product of two representations on the crossing strands:

$$T_1 \otimes T_2 = \sum Q$$

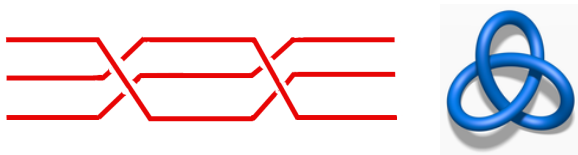
Since \mathcal{R} -matrix commutes with coproduct, it acts proportionally to a unity operator on such irreducible representations with the eigenvalue

$$\lambda_Q = q^{\varkappa_Q}$$

where \varkappa_Q is an eigenvalue of a second Casimir

Braid presentation of knots

The generalization of this approach is to study braid description of the knot:



All the operators (\mathcal{R} -matrices) then act on the product of all representations in the braid, rather than just on a pair of strands.

This means that \mathcal{R} -matrices for all representations Q_i from $T_1 \otimes T_2 \otimes T_3 \otimes \dots = \sum Q_i$ are needed.

The polynomial is given as a weighted trace of the product of \mathcal{R} -matrices:

$$H = \text{Tr}_q \prod \mathcal{R}_i$$

3-strand braids

The problem of calculating braid matrices has efficient solutions in fact only for 3-strand braids.

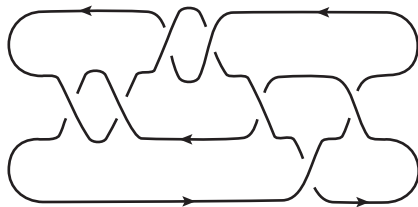
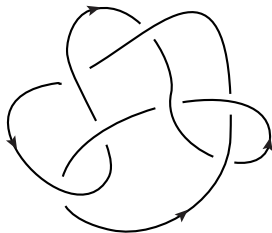
There only two types of matrices are needed – diagonal \mathcal{R} and Racah matrices U . Then needed \mathcal{R} -matrices are

$$\mathcal{R}_1 = \mathcal{R} \quad \mathcal{R}_2 = URU^\dagger$$

There are plenty of properties which simplify calculations of these matrices (e.g. eigenvalue hypothesis) and they are known for lots of representations such as **all symmetric and antisymmetric** representations and **all representations up to the size four** (see e.g. knotbook.org).

Two-bridge Knots

Different idea is used for **Two-bridge Knots**, which are made from 4-strand braid:



Due to the chosen closure from all the representations Q from $T_1 \otimes T_2 \otimes T_3 \otimes T_4 = \sum Q$ only the trivial one $Q = \emptyset$ remains.

Two-bridge Knots elements

This leads to the fact that only four distinct elements needed to calculate any 2-bridge knot (rather than two for each irreducible representation as was for 3-strand knots):

- 2 \mathcal{R} -matrices

$$T = \mathcal{R}_{RR} = \mathcal{R}_{\bar{R}\bar{R}} \quad \text{and} \quad \bar{T} = \mathcal{R}_{R\bar{R}} = \mathcal{R}_{\bar{R}R}$$

- 2 Racah matrices

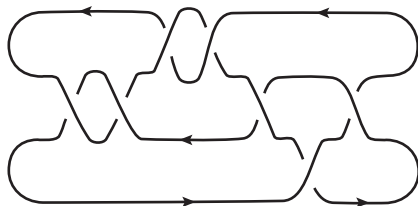
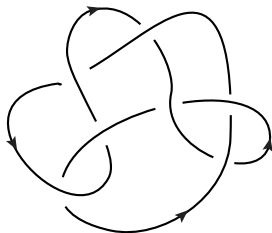
$$S = S \begin{pmatrix} R & R \\ \bar{R} & \bar{R} \end{pmatrix} = S \begin{pmatrix} \bar{R} & \bar{R} \\ R & R \end{pmatrix} = \left(S \begin{pmatrix} \bar{R} & R \\ \bar{R} & R \end{pmatrix} \right)^\dagger = \left(S \begin{pmatrix} R & \bar{R} \\ R & \bar{R} \end{pmatrix} \right)^\dagger$$

and

$$\bar{S} = S \begin{pmatrix} \bar{R} & R \\ R & \bar{R} \end{pmatrix} = S \begin{pmatrix} R & \bar{R} \\ \bar{R} & R \end{pmatrix}$$

Two-bridge knot answer

Index of the matrices corresponds to the irreducible representation coming from the product of two strands, because it uniquely defines that in the product of other two strands conjugate representation appears.

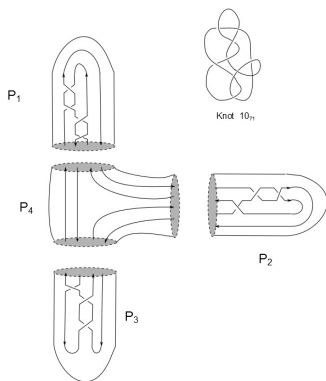


The answer for this one is given by $H_Q^K = d_Q^{-1} B_{00}$ with B equal to

$$B = \bar{S} \bar{T}^2 \bar{S} \bar{T}^{-2} \bar{S} \bar{T} S T^{-1} S^\dagger \bar{T} \bar{S}$$

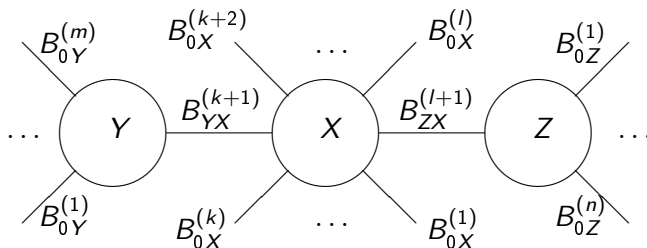
Arborescent knots

If one takes building blocks made from two-bridge knots, then these building blocks connected with each other form **arborescent** knots.



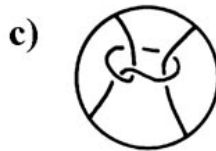
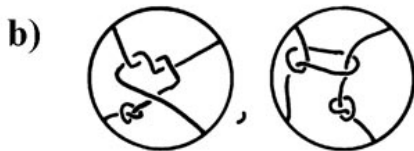
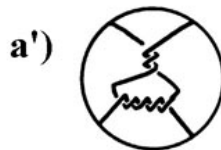
They can be even bigger with many branches. Here B_{YX} corresponds to two-bridge blocks with two open ends, B_{0X} to blocks with one open end and vertices X correspond to summing over all representations in the tensor product of two strands, connecting all the blocks.

For this the property dependence of the representations in two pairs of strands plays crucial role because it means that there is one and the same representation along the whole vertex.



Tangles

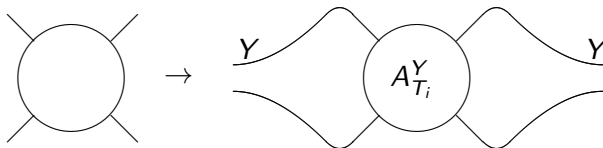
Tangle is an element of a knot, which is cut out by a 3-sphere with cutting made only in four points.



Tangle blocks

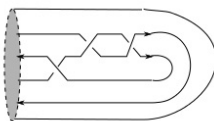
From arborescent calculus arise the idea of tangle blocks.

Tangle block means that we look at a tangle as an operator $A_{T_i}^Y$ acting in the space of unreducible representations in the product of two strands.



Two-bridge block as a tangle block

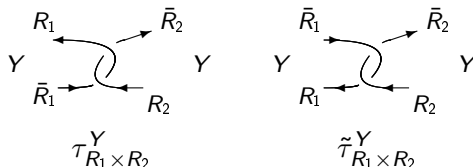
One of examples of a tangle is a two-strand block with one open end.



It is at quite complex tangle block, because it consists of many elements inside. However we know how to calculate it from two-bridge knots

$$A^Y = B_0 Y$$

Lock Block



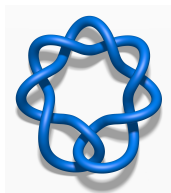
One of simplest tangles is a Lock Block. It is a particular example of two-bridge block and is given by

$$\tau_R^Y = \frac{D_R}{\sqrt{D_Y}} (\bar{S} \bar{T}^2 \bar{S})_{\emptyset Y}, \quad \tilde{\tau}_R^Y = \frac{D_R}{\sqrt{D_Y}} (S^\dagger T^2 S)_{\emptyset Y}$$

$$\tau_{[1]}^\emptyset = A \frac{A^2 q^2 - q^4 + q^2 - 1}{\{q\} q^2}, \quad \tau_{[1]}^{\text{adj}} = -A \{q\}$$

Uses of Lock Block

Lock Block though very simple allow to find two big classes of knots.
The first class are twist knots:

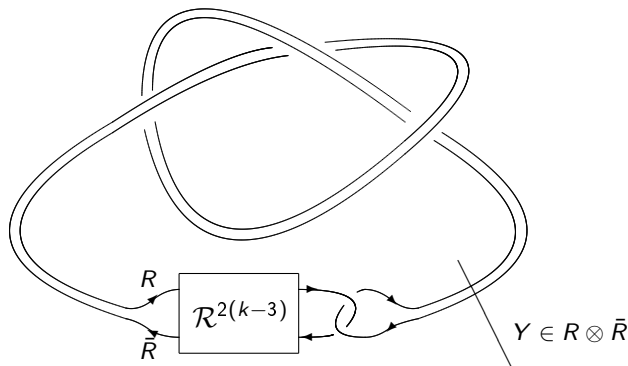


They are obtained as a product of Lock block and a braid:

$$H_R^{\text{Tw}_{2k}} = \sum_Y \tau_R^Y \lambda_R^{2(k-2)}$$

Whitehead double

The second class are Whitehead Doubles. They are constructed by placing a twist knot along any other knot \mathcal{K} and thus are satellites of knot \mathcal{K} .



Whitehead Double answer

The answer for Whitehead Double can be obtained quite easily from tangle formalism.

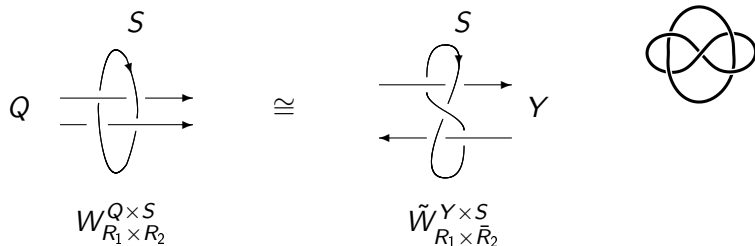
It is represented as a product of Lock block, a twisted braid and a knot itself:

$$H_R^{S_k(\mathcal{K})} = \frac{1}{D_R} \sum_{Y \in R \otimes \bar{R}} \mathcal{H}_Y^{\mathcal{K}} \cdot \lambda_Y^{2(k+w^{\mathcal{K}'})} \cdot \tau_R^Y$$

Thus knowing polynomial of a knot in a representation $Y \in R \otimes \bar{R}$, answer for Whitehead Double can be immediately calculated.

Whitehead block

More complicated Tangle block which cannot be constructed from two-bridge blocks is a Whitehead block

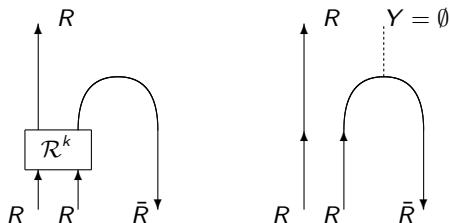


It can be calculated using some known examples and the result is

$$W_{[1] \times [1]}^{[2] \times [1]} = \frac{A^2 q^4 - q^6 - A^2 q^2 + 2q^4 + A^2 - 2q^2}{Aq^2 \{A\}}$$

Trident Block

One can even consider tangle blocks of different type – with different number of lines on two sides. One example of such is a trident block:



It is an operator acting between two different spaces: $\mathcal{T}_{R, R \otimes Y}^{\mathcal{K}_2}$

$$\mathcal{T}_{[1], [1] \otimes \emptyset}^{(k)} = \frac{A^k}{\sqrt{D_{[1]}}} \cdot \frac{q^{-k} \{Aq\} + (-q)^k \{A/q\}}{\{q^2\}}$$

Conclusion

- A knot can be cut into tangles with corresponding blocks and this provides a much wider set of described knots.
- One can even join tangle approach and braids by inserting tangles instead of \mathcal{R} -matrices.
- Interesting question is whether it is possible to construct universal basis of tangles which allows to describe all or most of knots.

THANK YOU FOR YOUR ATTENTION!

Chern-Simons theory

- 3-dimensional topological gauge theory - Chern-Simons theory

- $S_{CS} = \frac{k}{4\pi} \int \text{Tr} (\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A})$

- Wilson-loop averages:

$$H_Q^{\mathcal{K}}(A, q) = \left\langle \text{Tr}_{Q_1} P \exp \left(\oint_{\mathcal{K}_1} \mathcal{A} \right) \dots \text{Tr}_{Q_n} P \exp \left(\oint_{\mathcal{K}_n} \mathcal{A} \right) \right\rangle_{CS(N, q)}$$

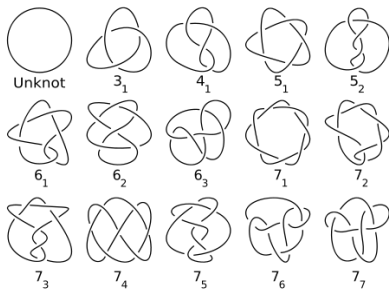
$Q_1 \dots Q_n$ are representations of the group $SU(N)$

$\mathcal{K} = \mathcal{K}_1 \cup \dots \cup \mathcal{K}_n$ is a collection of disjoint contours (i.e. a link)

$$q = \exp\left(\frac{2\pi i}{k+N}\right) \quad A = q^N$$

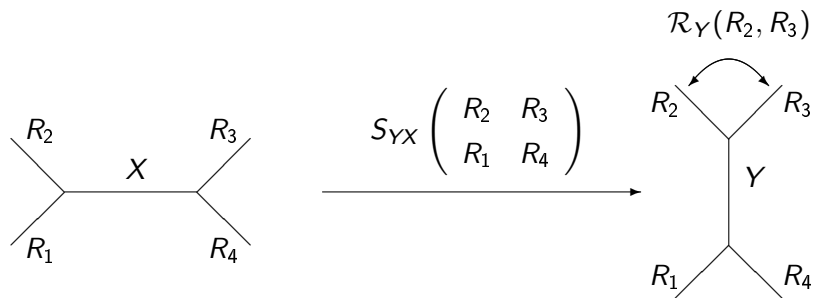
Knot theory

Wilson-loop averages of the Chern-Simons theory are equal to the (colored) knot invariants $H_Q^K(A, q)$ with coloring of representation Q . (For $SU(N)$ H_Q^K are called HOMFLY-PT polynomials).



Two-bridge Knots elements

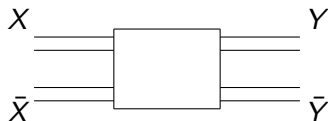
The elements from which HOMFLY of 2-bridge knots can be constructed are \mathcal{R} -matrices and Racah matrices (S_{YX}). But due to the form of the knot these Racah-matrices are for quite simple sets of representations:



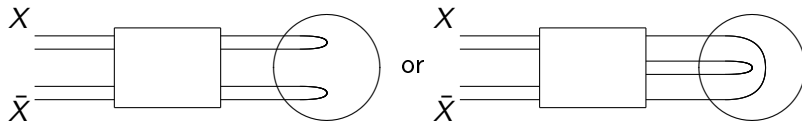
With all four representations R_1 , R_2 , R_3 and R_4 equal to either R or \bar{R} .

Propagators and Fingers

Propagators are just 2-bridge knots with open ends:

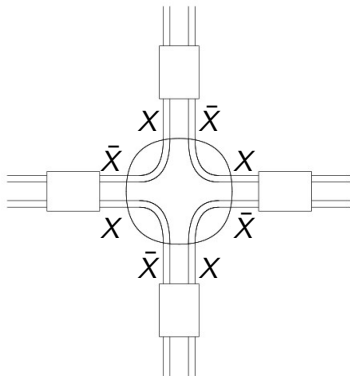


If one of the endings remains closed it becomes a “finger” or a “leaf” instead:

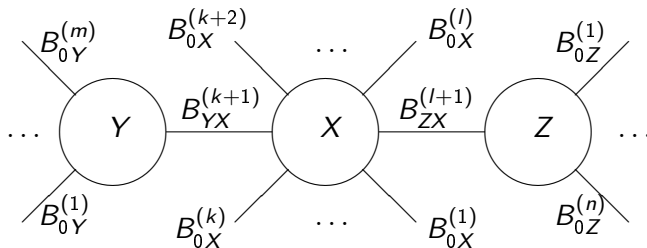


Vertex

Vertex describes summation over the corresponding index X :



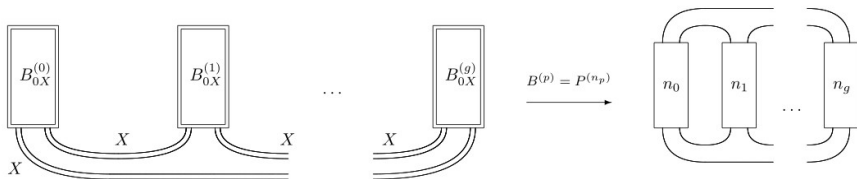
Effective theory structure



The answer is given by:

$$\begin{aligned}
 H = & \sum_{X,Y,Z} d_X d_Y d_Z \cdot \text{Tr}_{\mu_Y} \text{Tr}_{\mu_Z} \\
 & \left\{ \text{Tr}_{\mu_X} \left(\prod_{\alpha=1}^k B_{0X}^{(\alpha)} \cdot B_{YX}^{(k+1)} \prod_{\beta=k+2}^l B_{0X}^{(\beta)} \cdot B_{ZX}^{(l+1)} \right) \right\} \cdot \\
 & \cdot \left(\prod_{\gamma=1}^m B_{0Y}^{(\gamma)} \right) \cdot \left(\prod_{\delta=1}^n B_{0Z}^{(\delta)} \right)
 \end{aligned}$$

Example: Pretzel Knots



One of the simplest examples of arborescent knots are **Pretzel knots**. This is an example with just one vertex and several simple “leaves”. The answer for such knots is given by a very simple formula:

$$H_Q^{Pr(n_0, \dots, n_g)} = \frac{1}{d_Q} \sum_X d_X \prod_{i=0}^g \frac{(\bar{S}^\dagger \bar{T}^{n_i} S)_{0X}}{S_{0X}}$$

Lagrangian description

One can try to construct a Lagrangian description of this effective field theory. There three types of states/fields in this theory:



And three corresponding conjugate fields:



The vacuum transitions are provided by $\bar{J}\varphi_0$ and $J\phi_0$

Quadratic terms

Local quadratic terms of this Lagrangian are provided by \mathcal{R} -matrices and have the form:

$$\sigma_X T_X^n \sigma_X, \quad \varphi_X \bar{T}_X^{2n} \varphi_X, \quad \phi_X \bar{T}_X^{2n} \phi_X, \quad \varphi_X \bar{T}_X^{2n-1} \phi_X, \quad \phi_X \bar{T}_X^{2n-1} \varphi_X$$

There are also corresponding conjugate terms.

Non-local quadratic terms are provided by Racah matrices:

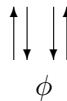
$$\sigma_X^* S_{XY}^\dagger \sigma_Y, \quad \phi_X^* S_{XY} \phi_Y, \quad \varphi_X^* \bar{S}_{XY} \varphi_Y$$

Vertices

Vertex terms in the Lagrangian can in fact be of any degree. E.g. the cubic ones have the form:

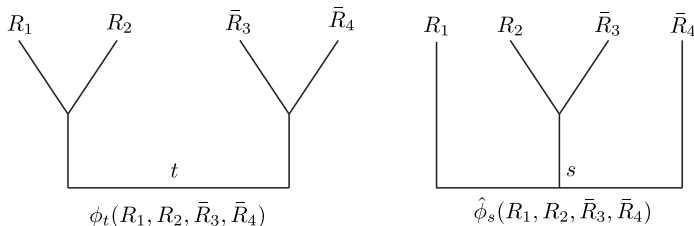
$$\Gamma^{(1)} \sim \sigma_X^3, \quad \Gamma^{(2)} \sim \varphi_X^3, \quad \Gamma^{(3)} \sim \phi_X^2 \varphi_X$$

Others are topologically not allowed which can be seen from the form of the fields:



WZNW CFT and 2-bridge knots

WZNW conformal theory is closely related to the knot theory. In particular, 2-bridge knots describe transformations of the WZNW conformal block.



Dimensions of the fields are in one-to-one correspondence with the representations on the strands in the 2-bridge knot, while the Racah coefficients correspond to the conformal block modular transformations.