

Solitons in the SU(3) Faddeev-Niemi Model

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In collaboration with Nobuyuki Sawado (TUS)

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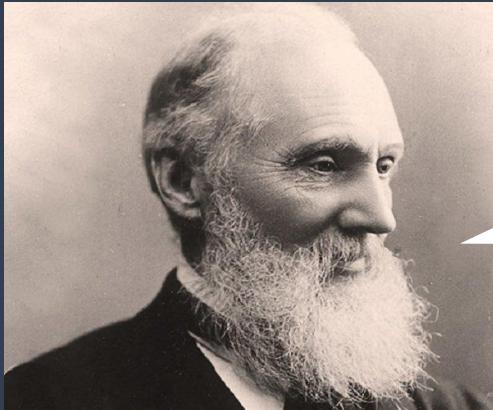
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2. Hopfions in the $SU(2)$ Faddeev-Niemi model
3. 2D instantons in the F_2 nonlinear σ -model
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5. Summary & Outlook

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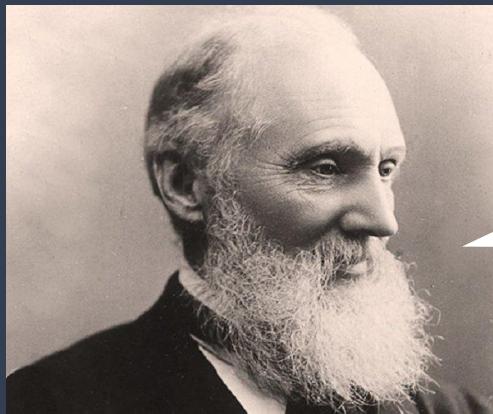
Knots as particles



*Atoms are knotted vortex tubes
in ether.*

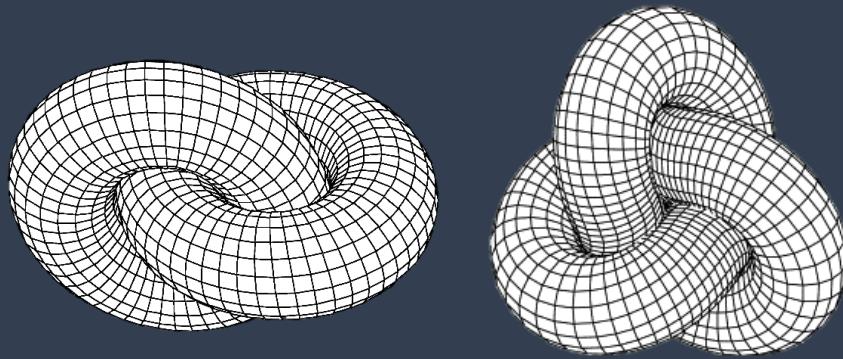
Lord Kelvin

Knots as particles

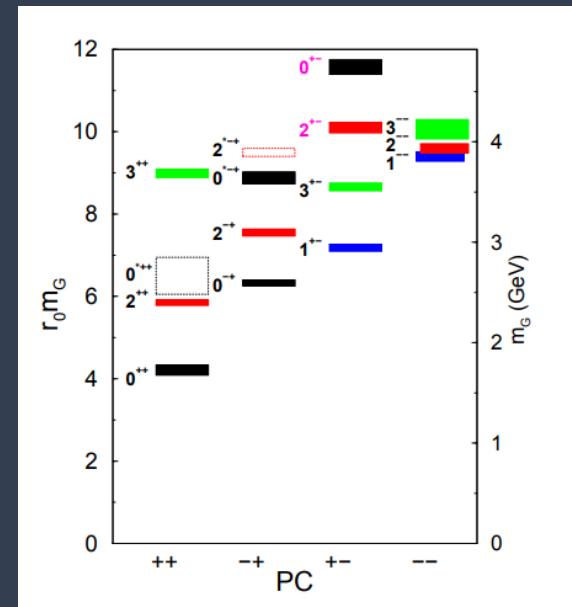


*Atoms are knotted vortex tubes
in ether.*

Lord Kelvin



The mass spectrum of glueballs



R. V. Buniy, T. W. Kephart,
“A model of glueballs” PLB **576**, 127

C. J. Morningstar and M. J. Peardon
PRD **60**, 034509 (1999)

SU(2) Yang-Mills fields in the IR limit

L. D. Faddeev & A. J. Niemi, PRL **82**, 1624 (1999)

QCD in UV limit ... asymptotic freedom
in IR limit ... color confinement

Very different phases!

→ In the IR limit with monopole condensation, some other order parameter could play a central role rather than the gauge field A_μ .

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Cho-Faddeev-Niemi (CFN) decomposition

$$A_\mu^a = C_\mu n_a + \varepsilon^{abc} \partial_\mu n_b n_c + \rho \partial_\mu n_a + \sigma \varepsilon^{abc} \partial_\mu n_b n_c$$

C_μ : Abelian gauge field, $\vec{n} = (n_1, n_2, n_3)$ with $|\vec{n}| = 1$,
 ρ, σ : scalar field

If $\vec{n} = \frac{\vec{r}}{r}$, $C_\mu = \rho = \sigma = 0$, it gives the Wu-Yang monopole.

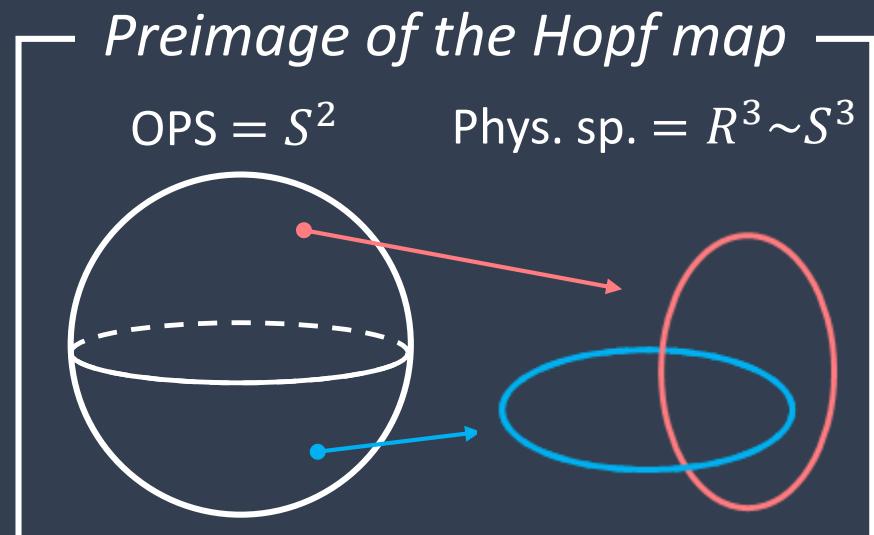
Effective model in confinement phase

In the confinement phase $\langle \phi \rangle = 0$ where $\phi = \rho + i\sigma$, we get
SU(2) Faddeev-Niemi model

$$S_{\text{eff}} = \int d^4x \left\{ \Lambda^2 (\partial_\mu \vec{n})^2 + (\partial_\mu \vec{n} \times \partial_\nu \vec{n})^2 \right\}$$

For static configurations,
 \vec{n} defines a mapping $S^3 \rightarrow S^2$.

$\pi_3(S^2) = \mathbb{Z}$  ***Knot solitons***



CFN decomposition of $SU(3)$ gauge fields

Decomposition L. D. Faddeev and A. J. Niemi, PLB **449**, 214 (1999)

$$A_\mu = C_\mu^a \mathbf{n}_a + i[\mathbf{n}_a, \partial_\mu \mathbf{n}_a] + \rho_{ab} \{ \mathbf{n}_a, \partial_\mu \mathbf{n}_b \} + i\sigma_{ab} [\mathbf{n}_a, \partial_\mu \mathbf{n}_b]$$

$$\mathbf{n}_a = U h_a U^\dagger \quad U \in SU(3)$$

$SU(3)$ Faddeev-Niemi model

$$S_{\text{eff}} = \sum_{a=1}^2 \int d^4x \left\{ M^2 \text{Tr}(\partial_\mu \mathbf{n}_a \partial^\mu \mathbf{n}_a) + \frac{1}{e^2} F_{\mu\nu}^a F^{a\mu\nu} \right\}$$

$$F_{\mu\nu}^a = -\frac{i}{2} \text{Tr}(\mathbf{n}_a [\partial_\mu \mathbf{n}_b, \partial_\nu \mathbf{n}_b])$$

$$\text{OPS} = SU(3)/U(1)^2 = F_2$$

$$\text{Cf. } S^2 = SU(2)/U(1)$$

$$\pi_3(F_2) = \mathbb{Z} \quad \rightarrow \quad \textbf{\textit{Knot solitons}}$$

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The $SU(2)$ Faddeev-Niemi model

J. Gladikowski & M. Hellmund, PRD **56**, 5194 (1997)

The static energy

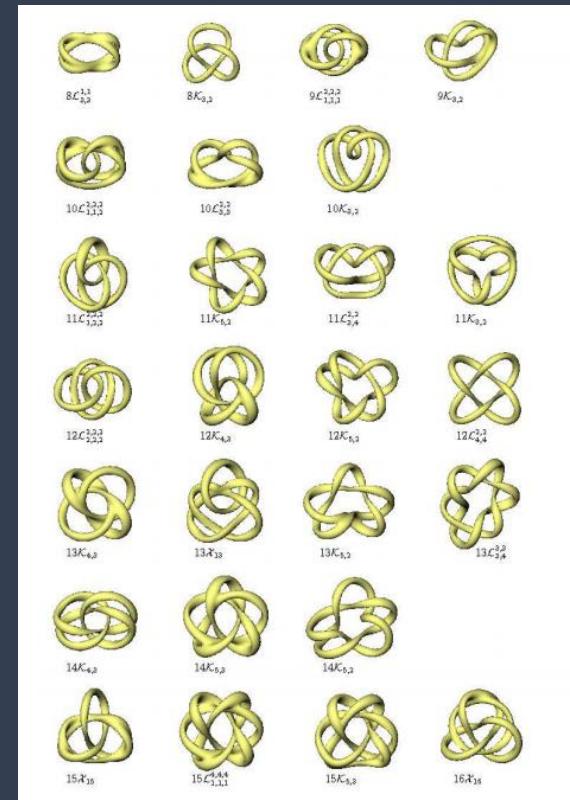
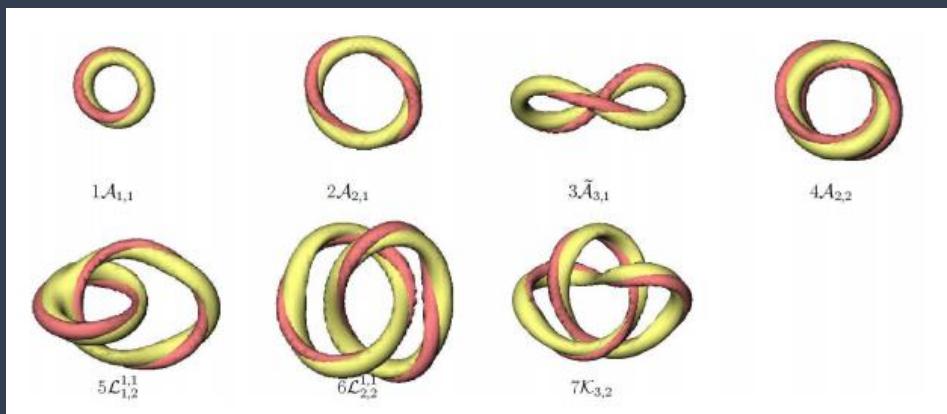
L. D. Faddeev & A. J. Niemi, Nature **387**, 58 (1997)

$$E = \int d^3x \left\{ M^2 (\partial_i \vec{n})^2 + \frac{1}{e^2} (\partial_i \vec{n} \times \partial_j \vec{n})^2 \right\}$$

The Hopfions

P. Sutcliffe, Proc. Roy. Soc. Lond. A **463**, 3001 (2007)

R. A. Battye & P. M. Sutcliffe, PRL **81**, 4798 (1998)



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Stereographic projection $S^2 \rightarrow CP^1$

$$\vec{n} = \frac{1}{\Delta} (u + u^*, -i(u - u^*), |u|^2 - 1) \quad \Delta = 1 + |u|^2$$

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At the end of my talk,
we see the e.o.m. of the $SU(3)$ model reduces to
that of the $SU(2)$ case with the stereographic projection.

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The F_2 nonlinear σ -model

The action density

$$\mathcal{S} = |Z_A^\dagger \partial_\mu Z_B|^2 + |Z_B^\dagger \partial_\mu Z_C|^2 + |Z_C^\dagger \partial_\mu Z_A|^2$$

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↓

$$D_\mu^{(a)} \equiv \partial_\mu - (Z_a^\dagger \partial_\mu Z_a)$$

Copies of the CP^2 NL σ -model

A. D'Adda et.al., NPB 146, 63 (1978)

$$S = \int d^2x \left(|D_\mu^{(A)} Z_A|^2 + |D_\mu^{(C)} Z_C|^2 - |Z_C^\dagger \partial_\mu Z_A|^2 \right)$$

The F_2 nonlinear σ -model

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$$S = \int d^2x \left(|D_\mu^{(A)} Z_A|^2 + |D_\mu^{(C)} Z_C|^2 - |Z_C^\dagger \partial_\mu Z_A|^2 \right)$$

$$\geq 2\pi(|N_A| + |N_C|) - \int d^2x |Z_C^\dagger \partial_\mu Z_A|^2 \quad \text{Non topological !}$$

Winding number of a map $S^2 \rightarrow CP^2$

$$N_a = \frac{i}{2\pi} \int d^2x \varepsilon^{\mu\nu} \left(D_\mu^{(a)} Z_a \right)^\dagger D_\nu^{(a)} Z_a \quad N_A + N_B + N_C = 0$$

Parametrization

Isomorphism $SU(3)/U(1)^2 \cong SL(3, \mathbb{C})/B_+$

$$\begin{pmatrix} 1 & 0 & 0 \\ u_1 & 1 & 0 \\ u_2 & u_3 & 1 \end{pmatrix} = (c_1, c_2, c_3) \in SL(3, \mathbb{C}) \quad 6 \text{ DOF} = \dim(F_2)$$



*Gram-Schmidt
orthogonalization process*

$$U = (Z_A, Z_B, Z_C) \in SU(3)$$

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**Gram-Schmidt
orthogonalization process**

$$U = (Z_A, Z_B, Z_C) \in SU(3)$$

$$Z_A = \frac{1}{\sqrt{\Delta_1}} \begin{pmatrix} 1 \\ u_1 \\ u_2 \end{pmatrix} \quad Z_B = \frac{1}{\sqrt{\Delta_1 \Delta_2}} \begin{pmatrix} -u_1^* - u_2^* u_3 \\ 1 - u_1 u_2^* u_3 + |u_2|^2 \\ -u_1^* u_2 + u_3 + u_3 |u_1|^2 \end{pmatrix} \quad Z_C = \frac{1}{\sqrt{\Delta_2}} \begin{pmatrix} u_1^* u_3^* - u_2^* \\ -u_3^* \\ 1 \end{pmatrix}$$

$$\Delta_1 = 1 + |u_1|^2 + |u_2|^2$$

$$\Delta_2 = 1 + |u_3|^2 + |u_1 u_3 - u_2|^2$$

The 2D instantons

The BPS-like bound

$$S \geq 2\pi(|N_A| + |N_C|) - \int d^2x |Z_C^\dagger \partial_\mu Z_A|^2$$

↑ satisfied if and only if $u_j = u_j(x_1 \pm ix_2)$ for $j = 1, 2, 3$.

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Stationary points in the bound correspond to the instantons.

$$-\pi(|N_A| + |N_C|) \leq -\int d^2x |Z_C^\dagger \partial_\mu Z_A|^2 \leq 0$$

$$u_1 = u_3 = 0$$

$$u_2 = u_2(x_1 \pm ix_2)$$

H. T. Ueda, Y. Akagi & N. Shannon,
PRA. **93**, 021606(R) (2016)

Embedding type

The 2D instantons

The BPS-like bound

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H. T. Ueda, Y. Akagi & N. Shannon,
PRA. **93**, 021606(R) (2016)

$$u_k = u_k(x_1 \pm ix_2) \quad k = 1, 2$$
$$u_3 \partial_\mu u_1 - \partial_\mu u_2 = 0$$

Y.A. & N. Sawado,
PRD **97**, 065012 (2018)

Embedding type

Genuine type

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The $SU(3)$ Faddeev-Niemi model

The Lagrangian

$$\mathcal{L} = \sum_{a=1}^2 M^2 \text{Tr}(\partial_\mu \mathfrak{n}_a \partial^\mu \mathfrak{n}_a) - \frac{1}{e^2} F_{\mu\nu}^a F^{a\mu\nu}$$

For simplicity,
we set $M = e = 1$.

$$F_{\mu\nu}^a = -\frac{i}{2} \text{Tr}(\mathfrak{n}_a [\partial_\mu \mathfrak{n}_b, \partial_\nu n_b])$$

$$\mathfrak{n}_a = U h_a U^\dagger \quad U = (Z_A, Z_B, Z_C) \in SU(3)$$

The Cartan-Weyl basis

$$h_1 = \lambda_3/\sqrt{2}, \quad h_2 = \lambda_8/\sqrt{2}$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad e_{-p} = (e_p)^T$$

Equations of motion

Cartan decomposition $U^\dagger \partial_\mu U = iA_\mu^a h_a + i\mathcal{J}_\mu^p e_p$

The EL eq. is equivalent to conservation of the Noether current \mathcal{K}_μ associated to $U \rightarrow gU$, $g \in SU(3)$.

$$\partial^\mu \mathcal{K}_\mu = 0 \quad \xrightarrow{\mathcal{K}_\mu = U \mathcal{B}_\mu U^\dagger} \quad \partial_\mu \mathcal{B}^\mu + [U^\dagger \partial^\mu U, \mathcal{B}_\mu] = 0$$
$$\mathcal{B}_\mu = i \sum_p (\mathcal{J}_\mu^p - i\alpha_a^p F_{\mu\nu}^a \mathcal{J}^{p\nu}) e_p$$

Equations of motion

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The equations of motion

$$\begin{aligned} \partial^\mu (\mathcal{J}_\mu^p - iG_{\mu\nu}^p \mathcal{J}^{p\nu}) + iR^{p\mu} (\mathcal{J}_\mu^p - iG_{\mu\nu}^p \mathcal{J}^{p\nu}) \\ + G^{p\mu\nu} \mathcal{J}_\mu^{-p-1} \mathcal{J}_\nu^{-p+1} = 0 \quad \forall p = 1, 2, 3 \end{aligned}$$

$$R_\mu^p = \alpha_a^p A_\mu^a, \quad G_{\mu\nu}^p = \alpha_a^p F_{\mu\nu}^a \quad \alpha_a^p = \text{Tr}(e_{-p}[h_a, e_p])$$

Trivial embedding

The embedding can be realized if two of the scalars vanish.

$$\begin{aligned} u_1 = u_3 = 0, \quad & \Rightarrow \quad U = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} 1 & 0 & -u^* \\ 0 & \sqrt{\Delta} & 0 \\ u & 0 & 1 \end{pmatrix} \quad \Delta = 1 + |u|^2 \\ u_2 = u \end{aligned}$$

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We have $\mathcal{J}_\mu^1 = \mathcal{J}_\mu^3 = 0$, $\mathcal{J}_\mu^2 = \frac{i}{\Delta} \partial_\mu u$

The e.o.m for $p = 1, 3$ are automatically satisfied and for $p = 2$ reduces to

$$\partial^\mu [\partial_\mu u - iG_{\mu\nu}\partial^\nu u] + (iR_\mu - \partial_\mu \log \Delta)(\partial^\mu u - iG^{\mu\nu}\partial_\nu u) = 0$$

$$R_\mu = \frac{i}{\Delta} (u^* \partial_\mu u - u \partial_\mu u^*) \quad G_{\mu\nu} = -\frac{2i}{\Delta^2} (\partial_\mu u \partial_\nu u^* - \partial_\mu u^* \partial_\nu u)$$

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The EL eq. of the $SU(2)$ Faddeev-Niemi model

Genuine solution

Our strategy

(i) We impose $\mathcal{J}_\mu^2 \propto u_3 \partial_\mu u_1 - \partial_\mu u_2 = 0$.

→ $u_2 = f(u_1), u_3 = f'(u_1)$

(ii) EL eq. for $p = 1, 3$ are proportional to each other.

→ $(1 + |u_1|^2 + |u_2|^2)/(1 + |u_3|^2 + |u_1 u_3 - u_2|^2) = \text{const.}$

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$$\begin{aligned} u_1 &= u_3 = \sqrt{2}u, \\ u_2 &= -u^2 \end{aligned} \quad \Rightarrow \quad U = \frac{1}{\Delta} \begin{pmatrix} 1 & -\sqrt{2}u^* & -u^{*2} \\ \sqrt{2}u & 1 - |u|^2 & \sqrt{2}u^* \\ -u^2 & -\sqrt{2}u & 1 \end{pmatrix}$$

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The EL eq. of the $SU(2)$ Faddeev-Niemi model !!

Summary

We confirmed the existence of Hopfions
in the $SU(3)$ Faddeev-Niemi model.

The model is an effective model of the $SU(3)$ pure YM theory.

We found an ansatz of nonembedding type which the e.o.m.
reduces to the $SU(2)$ Faddeev-Niemi model.

Outlook

Stability of the Hopfions

Implication of the constraint

Estimation of mass spectrum of glueballs

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Thank you for your attention!

Reformulation

Cartan decomposition of the Murer-Cartan form

$$U^\dagger \partial_\mu U = iA_\mu^a h_a + i\mathcal{J}_\mu^p e_p \quad p = \pm 1, \pm 2, \pm 3.$$

Under the gauge transformation $U \rightarrow U \exp(i\theta^a h_a)$,

$$A_\mu^a \rightarrow A_\mu^a + \partial_\mu \theta^a, \quad \mathcal{J}_\mu^p \rightarrow \mathcal{J}_\mu^p e^{-i\theta^a \alpha_a^p}$$

The energy functional

$$E = \int d^3x \sum_{q=1}^3 \left[\mathcal{J}_i^q \mathcal{J}_i^{-q} - \frac{1}{4} \left(\mathcal{J}_{[i}^q \mathcal{J}_{j]}^{-q} - \mathcal{J}_{[i}^{q+1} \mathcal{J}_{j]}^{-(q+1)} \right)^2 \right]$$
$$\mathcal{J}_{[i}^q \mathcal{J}_{j]}^{-q} \equiv \mathcal{J}_i^q \mathcal{J}_j^{-q} - \mathcal{J}_j^q \mathcal{J}_i^{-q}$$

The energy represents the gauge fixing functional
for a *nonlinear maximal Abelian gauge*.

Parametrization

$$\begin{pmatrix} \chi_1 & 0 & 0 \\ \chi_2 & \chi_4 & 0 \\ \chi_3 & \chi_5 & (\chi_1\chi_4)^{-1} \end{pmatrix} = (c_1, c_2, c_3) \quad \chi_1 \neq 0, \chi_4 \neq 0 \quad 10 \text{ dof}$$

Gramm-Schmit orthogonalization process

$$\chi_2/\chi_1 = u_1, \chi_3/\chi_1 = u_2, \chi_5/\chi_4 = u_3 \quad \vartheta_i = \arg(\chi_i)$$

$$U = (Z_A e^{i\vartheta_1}, Z_B e^{i\vartheta_4}, Z_C e^{-i(\vartheta_1 + \vartheta_4)}) \quad 8 \text{ dof} = \dim(SU(3))$$

$$W = (Z_A, Z_B, Z_C) \quad 6 \text{ dof} = \dim(F_2)$$

$$Z_A = \frac{1}{\sqrt{\Delta_1}} \begin{pmatrix} 1 \\ u_1 \\ u_2 \end{pmatrix} \quad Z_B = \frac{1}{\sqrt{\Delta_1 \Delta_2}} \begin{pmatrix} -u_1^* - u_2^* u_3 \\ 1 - u_1 u_2^* u_3 + |u_2|^2 \\ -u_1^* u_2 + u_3 + u_3 |u_1|^2 \end{pmatrix} \quad Z_C = \frac{1}{\sqrt{\Delta_2}} \begin{pmatrix} u_1^* u_3^* - u_2^* \\ -u_3^* \\ 1 \end{pmatrix}$$

$$\Delta_1 = 1 + |u_1|^2 + |u_2|^2 \quad \Delta_2 = 1 + |u_3|^2 + |u_1 u_3 - u_2|^2$$

The Hopf invariant

M. Kisielowski, J. Phys. A **49**, 17, 175206 (2016)

Exact sequence

$$0 \rightarrow \cdots \rightarrow \pi_3(SU(3)) \rightarrow \pi_3(SU(3)/U(1)^2) \rightarrow \cdots \rightarrow 0$$

Isotropy

$$\pi_3(F_2) \cong \pi_3(SU(3))$$

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Hopf invariant

=Winding number of $U: S^3 \rightarrow SU(3)$

=CS term for the $SU(3)$ pure gauge R

$$H = \frac{1}{24\pi^2} \int \text{Tr}(U^\dagger dU)^3 = \frac{1}{16\pi^2} \int \text{Tr}(R \wedge dR) + \frac{2}{3} \text{Tr}(R \wedge R \wedge R)$$

$$R = U^\dagger dU$$

The Hopf invariant

$$H = \frac{1}{8\pi^2} \int d^3x \ \varepsilon^{ijk} A_i^a F_{jk}^a + \frac{1}{8\pi^2} \int d^3x \ \varepsilon^{ijk} \text{Im}(\mathcal{J}_i^1 \mathcal{J}_j^2 \mathcal{J}_k^3)$$

CFN decomposition of $SU(3)$ gauge fields

There are two options of the decomposition:

SB pattern	Order parameter space	Homotopy group
$SU(3) \rightarrow U(1)^2$	$F_2 = SU(3)/U(1)^2$	$\pi_3(F_2) = \mathbb{Z}$
$SU(3) \rightarrow U(2)$	$CP^2 = SU(3)/U(2)$	$\pi_3(CP^2) = 0$

Decomposition L. D. Faddeev and A. J. Niemi, PLB **449**, 214 (1999)

$$A_\mu = C_\mu^a \mathbf{n}_a + i[\mathbf{n}_a, \partial_\mu \mathbf{n}_a] + \rho_{ab} \{\mathbf{n}_a, \partial_\mu \mathbf{n}_b\} + i\sigma_{ab} [\mathbf{n}_a, \partial_\mu \mathbf{n}_b]$$

$SU(3)$ Faddeev-Niemi model

$$S_{\text{eff}} = \sum_{a=1}^2 \int d^4x \left\{ M^2 \text{Tr}(\partial_\mu \mathbf{n}_a \partial^\mu \mathbf{n}_a) + \frac{1}{e^2} F_{\mu\nu}^a F^{a\mu\nu} \right\}$$

$$\mathbf{n}_a = Uh_a U^\dagger \quad U \in SU(3)$$

$$F_{\mu\nu}^a = -\frac{i}{2} \text{Tr}(\mathbf{n}_a [\partial_\mu \mathbf{n}_b, \partial_\nu \mathbf{n}_b])$$

Effective models of the $SU(2)$ YM theory

(1) Coulomb phase

The original Yang-Mills action at low energy

(2) Higgs phase $\langle \phi \rangle \neq 0$ $\phi = \rho + i\sigma$

$$S_{\text{eff}} = \int d^4x \left\{ G_{\mu\nu}^2 + |D_\mu \phi|^2 + (1 - |\phi|^2)^2 \right\}$$

$$G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu, \quad D_\mu = \partial_\mu + iC_\mu \quad \textbf{Abelian Higgs model}$$

(3) Confinement phase $\langle \phi \rangle = 0$

$$S_{\text{eff}} = \int d^4x \left\{ \Lambda^2 (\partial_\mu \vec{n})^2 + (\partial_\mu \vec{n} \times \partial_\nu \vec{n})^2 \right\}$$

$SU(2)$ Faddeev-Niemi model

The $SU(2)$ Faddeev-Niemi model

J. Gladikowski & M. Hellmund, PRD **56**, 5194 (1997)

L. D. Faddeev & A. J. Niemi, Nature **387**, 58 (1997)

The static energy

$$E = \int d^3x \left\{ M^2 (\partial_i \vec{n})^2 + \frac{1}{e^2} (\partial_i \vec{n} \times \partial_j \vec{n})^2 \right\}$$

$\vec{n} \rightarrow \text{const. as } x \rightarrow \infty \Rightarrow \vec{n} : R^3 \sim S^3 \rightarrow S^2 \quad \text{Hopf map}$

$\pi_3(S^2) = \mathbb{Z}$ The top. charge can be defined.

Derrick's theorem

$$E = E_2 + E_4 \quad \xrightarrow{\text{Scaling } x \rightarrow \lambda x} E[\vec{n}; \lambda] = \lambda^{-1} E_2 + \lambda E_4$$

$$\partial_\lambda E[\vec{n}; 1] = -E_2 + E_4 = 0$$

If E_4 is absent,

$$\partial_\lambda^2 E[\vec{n}; 1] = 2E_2 > 0$$

there are no stable solitons.

The $SU(2)$ Faddeev-Niemi model

J. Gladikowski & M. Hellmund, PRD **56**, 5194 (1997)

The static energy

L. D. Faddeev & A. J. Niemi, Nature **387**, 58 (1997)

$$E = \int d^3x \left\{ M^2 (\partial_i \vec{n})^2 + \frac{1}{e^2} (\partial_i \vec{n} \times \partial_j \vec{n})^2 \right\}$$

Stereographic projection $S^2 \rightarrow CP^1$

$$\vec{n} = \frac{1}{\Delta} (u + u^*, -i(u - u^*), |u|^2 - 1) \quad \Delta = 1 + |u|^2$$

The complex scalar u is convenient to solve the EL eq..

But ..., the topological charge cannot be defined by \vec{n} nor u .

→ We need to introduce $Z = (Z_0, Z_1)^T$ where $u = Z_1/Z_0$.

Hopf invariant $\in \pi_3(S^2) = \mathbb{Z}$

$$H = \frac{1}{4\pi} \int d^3x \mathcal{A} \wedge d\mathcal{A} \quad \mathcal{A} = iZ^\dagger dZ$$

Applications in condensed matter physics

There are many suggestions that
knot solitons appear in condensed matter physics;

- Triplet superconductor
- Nematic liquid crystal

E. Babaev, PRL **88**, 177002 (2002)

P. J. Ackerman & I. I. Smalyukh,
PRX **7**, 011006 (2017)

SU(3) analogue

- Color superconductor
- Spin-nematic

} described by
the *SU(3)* Heisenberg model.

OPS of the *SU(3)* AFH model on the Cubic lattice = F_2

Continuum limit → The *SU(3)* Faddeev-Niemi (like) model