

QUARKS-2018

XXth International Seminar
on High Energy Physics

Structure of the UV Divergences in Maximally Supersymmetric Theories



Dmitry Kazakov



Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia

Moscow Institute of Physics and Technology, Dolgoprudny, Russia

in collaboration with A. Borlakov, D.Tolkachev and D.Vlasenko

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arXiv:1712.04348 [hep-th], arXiv:1804.08387 [hep-th]

Motivation

Maximal SYM

D=4 N=4

D=6 N=2

D=8 N=1

D=10 N=1

- ➊ Partial or total cancellation of UV divergences
(all bubble and triangle diagrams cancel)
- ➋ First UV divergent diagrams at $D=4+6/L$
- ➌ Conformal or dual conformal symmetry
- ➍ Common structure of the integrands

Bern, Dixon &Co 10
Drummond, Henn,
Korchemsky, Sokatchev 10
Arkani-Hamed 12

Motivation

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D=4 N=4

D=6 N=2

D=8 N=1

D=10 N=1

- Partial or total cancellation (all bubble and loop momenta)
 - First 1000 terms of the perturbative expansion can be obtained from 10dim superstring by compactification on a torus
- All of them can be obtained from 10dim superstring by compactification on a torus
- via conformal symmetry
- and the similar structure of the integrands

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Drummond, Henn,
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Arkani-Hamed 12

Motivation

Maximal SYM

D=4 N=4

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D=8 N=1

D=10 N=1

- Partial or total cancellation (all bubble and contact terms)
 - First 100 amplitudes were obtained by direct integration at 1 loop
- All of them can be obtained from 10dim superstring by compactification on a torus (cancel conformal symmetry and structure of the integrands)

D=4 N=8 Supergravity

- On-shell finite up to 8 loops
- Similar to higher dim SYM

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Motivation

Maximal SYM

D=4 N=4

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D=8 N=1

D=10 N=1

- Partial or total cancellation (all bubble and contact terms)
 - First 10 amplitudes were obtained by direct integration at D=10
- All of them can be obtained from 10dim superstring by compactification on a torus (cancel anomalies)
- On-shell conformal symmetry and structure of the integrands

D=4 N=8 Supergravity

- On-shell finite up to 8 loops
- Similar to higher dim SYM

Object: Helicity Amplitudes on mass shell with arbitrary number of legs and loops

The case: Planar limit $N_c \rightarrow \infty$, $g_{YM}^2 \rightarrow 0$ and $g_{YM}^2 N_c$ - fixed

The aim: to get all loop (exact) result

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Motivation

Maximal SYM

D=4 N=4

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- Partial or total cancellation (all bubble and contact terms)
 - First 100 amplitudes known
- All of them can be obtained from 10dim superstring by compactification on a torus
- and conformal symmetry
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Study of higher dim SYM gives insight into quantum gravity

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UV divergences in all Loops

Spinor-helicity formalism: S-matrix elements

D=4 N=4

No UV div

IR div on shell

D=6 N=2

UV div from 3 loops

No IR div

D=8 N=1

UV div from 1 loop

No IR div

D=10 N=1

UV div from 1 loop

No IR div

All these theories are non-renormalizable by power counting

The coupling g^2 has dimension $[g^2] = \frac{1}{M^{D-4}}$

The aim: to get all loop (exact) result for the leading (at least) divs

Perturbation Expansion for the 4-point Amplitudes for any D

$$A_4/A_4^{tree}$$

$$g^2 \quad st \quad \boxed{}$$

**No bubbles
No Triangles**

$$-g^6 \quad s^3 t \quad \boxed{} \quad : \quad s^2 t \quad \boxed{} \quad : \quad st^2 \quad \boxed{} \quad : \quad st^3 \quad \boxed{}$$

First UV div at $\epsilon = [6/(D-4)]$ loops

$$g^8 - s^4 t \begin{array}{|c|c|c|c|} \hline \end{array} + s^3 t \begin{array}{|c|c|c|} \hline \end{array} \begin{array}{|c|} \hline \end{array} - s^2 t \begin{array}{|c|} \hline \end{array} \begin{array}{|c|} \hline \end{array} \begin{array}{|c|} \hline \end{array} + st^4 + s^5 t \begin{array}{|c|c|c|} \hline \end{array} \begin{array}{|c|} \hline \end{array} + \dots$$

IR finite

$$g^{10} - \frac{s^5 t}{s^4 t} \begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline & & & & \\ \hline\end{array} + \frac{s^3 t}{s^4 t} \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline\end{array} + \frac{s t^5}{s^2 t} \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & + & \\ \hline & & \\ \hline & & \\ \hline\end{array} + s^4 t \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline & + & \\ \hline & & \\ \hline & & \\ \hline\end{array} + s^2 t \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & + & \\ \hline & & \\ \hline & & \\ \hline\end{array} + \dots$$

T. Dennen Yu-yin Huang 10 ,
S.Caron-Huot D.O'Connell 10

Universal expansion for any D in maximal SYM due to Dual conformal invariance

Leading Divergences from Generalized «Renormalization Group»

- In renormalizable theories the leading divergences can be found from the 1-loop term due to the renormalization group, in particular, for a single coupling theory the coefficient of $1/\epsilon^n$ in n loops is

$$\mathcal{R}'G = \sum_n \frac{a_n^{(n)}}{\epsilon^n} \quad a_n^{(n)} = (a_1^{(1)})^n$$

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- In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and R-operation

$$\mathcal{R}'G = 1 - \sum_{\gamma} K\mathcal{R}'_{\gamma} + \sum_{\gamma, \gamma'} K\mathcal{R}'_{\gamma} K\mathcal{R}'_{\gamma'} - \dots,$$

Leading Divergences from Generalized «Renormalization Group»

- In renormalizable theories the leading divergences can be found from the 1-loop term due to the renormalization group, in particular, for a single coupling theory the coefficient of $1/\epsilon^n$ in n loops is**

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- In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and R-operation**

$$\mathcal{R}'G = 1 - \sum_{\gamma} K\mathcal{R}'_{\gamma} + \sum_{\gamma, \gamma'} K\mathcal{R}'_{\gamma} K\mathcal{R}'_{\gamma'} - \dots,$$

$$\begin{aligned} \mathcal{R}'G_n = & \frac{A_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + \dots + \frac{A_1^{(n)}(\mu^2)^\epsilon}{\epsilon^n} \\ & + \frac{B_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^{n-1}} + \frac{B_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^{n-1}} + \dots + \frac{B_1^{(n)}(\mu^2)^\epsilon}{\epsilon^{n-1}} \end{aligned}$$

+ lower order terms

Leading pole 

SubLeading pole 

$A_1^{(n)}, B_1^{(n)}$ **1-loop graph**
 $B_2^{(n)}$ **2-loop graph**

SubLeading Divergences from Generalized «Renormalization Group»

- In non-renormalizable theories the leading divergences can be also found from 1-loop due to locality and R-operation

All terms like $(\log \mu^2)^m / \epsilon^k$ should cancel

$$A_n^{(n)} = (-1)^{n+1} \frac{A_1^{(n)}}{n},$$

$$B_n^{(n)} = (-1)^n \left(\frac{2}{n} B_2^{(n)} + \frac{n-2}{n} B_1^{(n)} \right)$$

Leading pole
from 1 loop
diagrams

SubLeading pole
from 2 loop
diagrams

$$\mathcal{K}\mathcal{R}'G_n = \sum_{k=1}^n \left(\frac{A_k^{(n)}}{\epsilon^n} + \frac{B_k^{(n)}}{\epsilon^{n-1}} \right) \equiv \frac{A_n^{(n)'} \epsilon^n}{\epsilon^n} + \frac{B_n^{(n)'} \epsilon^{n-1}}{\epsilon^{n-1}}.$$

Just like in
renormalizable
theories one can
deduce the
leading,
subleading, etc
divergences from
1, 2, etc diagrams

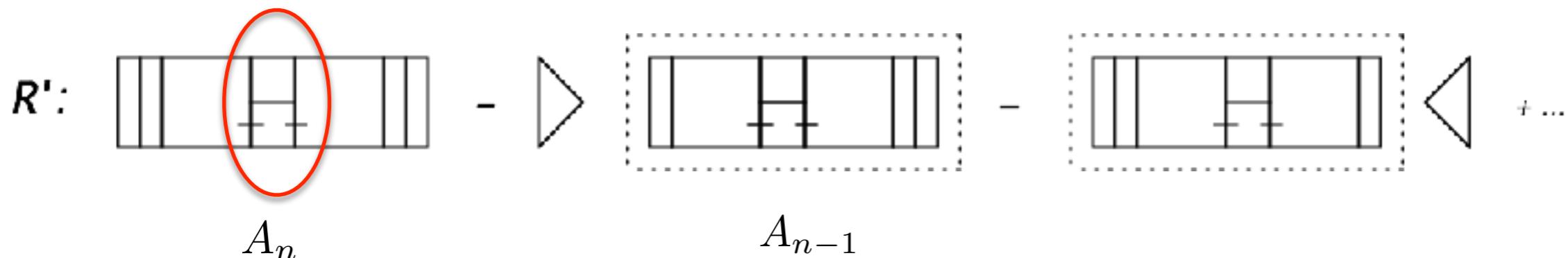
$$A_n^{(n)'} = (-1)^{n+1} A_n^{(n)} = \frac{A_1^{(n)}}{n},$$

$$B_n^{(n)'} = \left(\frac{2}{n(n-1)} B_2^{(n)} + \frac{2}{n} B_1^{(n)} \right)$$

R-operation and Recurrence Relation

D=6 N=2

Horizontal boxes + tennis court



$$nA_n = -A_{n-1} \quad \longrightarrow \quad A_n = (-1)^n \frac{2}{n!} \quad (-g^2 s)^n$$

Summation

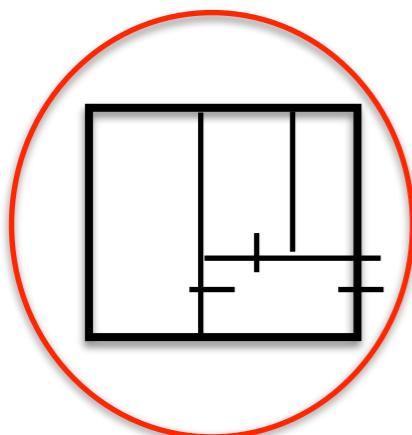
$$\Sigma_L = \sum_{n=3}^{\infty} A_n (-z)^n \quad z \equiv \frac{g^2 s}{\epsilon}$$

$$\boxed{\Sigma_L = \frac{2}{z^2} \left(e^z - 1 - z - \frac{z^2}{2} \right)}$$

$$\epsilon \rightarrow 0 \quad \Sigma_L \rightarrow \begin{cases} \infty & s > 0 \\ -1 & s < 0 \end{cases}$$

R-operation and Recurrence Relation

D=6 N=2



Horizontal boxes + double tennis court

$$nA_n^t = -\frac{1}{3}A_{n-1}^t, \quad nA_n^s = -A_{n-1}^s + \frac{1}{3}A_{n-1}^t$$

$$A_n^t = \frac{(-1)^n}{3^{n-3}} \frac{1}{n!}, \quad A_n^s = \frac{1}{2} \frac{(-1)^n}{3^{n-3}} \frac{1}{n!} - \frac{1}{2} (-1)^n \frac{1}{n!}$$

$$(-g^2 s)^{n-1} (-g^2 t) \quad (-g^2 s)^n$$

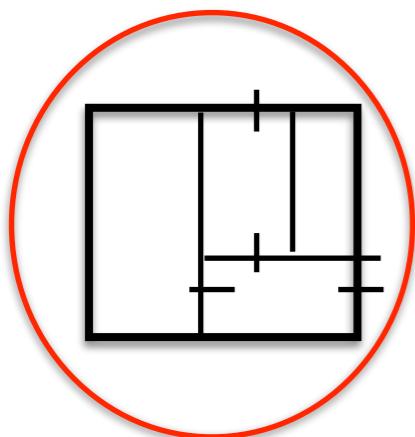
Summation

$$\Sigma_{L2} = \sum_{n=3}^{\infty} A_n^s (-z)^n + \frac{t}{s} A_n^t (-z)^n \quad z \equiv \frac{g^2 s}{\epsilon}$$

$$\Sigma_{L2} = \frac{1}{2s^2 z^2} \left[27(e^{z/3} - 1 - \frac{z}{3} - \frac{1}{2} \frac{z^2}{9} - \frac{1}{6} \frac{z^3}{27}) (1 + 2 \frac{t}{s}) - (e^z - 1 - sz - \frac{1}{2} z^2 - \frac{1}{6} z^3) \right]$$

R-operation and Recurrence Relation

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$$(-g^2 s)^{n-1} (-g^2 t)$$



$$nA_n^s = -A_{n-1}^s + \frac{1}{3}A_{n-1}^t$$

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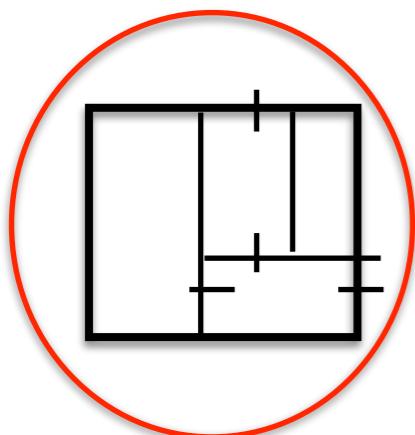
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- Similar relations one can get for all other series
- All of them have $1/n!$ behavior
- Number of these series group as $n!$

All loop Exact Recurrence Relation

D=6 N=2

s-channel term $S_n(s, t)$ **t-channel term** $T_n(s, t)$ $T_n(s, t) = S_n(t, s)$

Exact relation for ALL diagrams

$$nS_n(s, t) = -2s \int_0^1 dx \int_0^x dy (S_{n-1}(s, t') + T_{n-1}(s, t'))$$

$$\begin{aligned} n &\geq 4 \\ t' &= t(x-y) - sy \end{aligned}$$

$$S_3 = -s/3, \quad T_3 = -t/3$$

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Summation

$$\Sigma_k(s, t, z) = \sum_{n=k}^{\infty} (-z)^n S_n(s, t)$$

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Summation $\Sigma_k(s, t, z) = \sum_{n=k}^{\infty} (-z)^n S_n(s, t)$

Diff eqn $\frac{d}{dz} \Sigma_4(s, t, z) = 2s \int_0^1 dx \int_0^x dy (\Sigma_3(s, t', z) + \Sigma_3(t', s, z))|_{t'=xt+yu}$

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$$\Sigma_4(s, t, z) = \Sigma_3(s, t, z) + S_3(s, t)z^3 \quad \Sigma(s, t, z) = z^{-2}\Sigma_3(s, t, z)$$

All loop Exact Recurrence Relation

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Ladder diagrams (leading divs)

D=8 N=1

Horizontal boxes

$$A_n^{(n)} = s^{n-1} A_n$$

$$nA_n = -\frac{2}{4!}A_{n-1} + \frac{2}{5!} \sum_{k=1}^{n-2} A_k A_{n-1-k}, \quad n \geq 3$$

$$A_1 = 1/6$$

1 loop box

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$$\Sigma_m(z) = \sum_{n=m}^{\infty} A_n (-z)^n$$

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$$\Sigma_A \equiv \Sigma_1$$

Diff eqn

$$\frac{d}{dz}\Sigma_A = -\frac{1}{3!} + \frac{2}{4!}\Sigma_A - \frac{2}{5!}\Sigma_A^2$$

$$z = g^2 s^2 / \epsilon$$

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$$\Sigma_A(z) = -\sqrt{5/3} \frac{4 \tan(z/(8\sqrt{15}))}{1 - \tan(z/(8\sqrt{15}))\sqrt{5/3}} = \sqrt{10} \frac{\sin(z/(8\sqrt{15}))}{\sin(z/(8\sqrt{15}) - z_0)}$$

Ladder diagrams (leading divs)

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Summation

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$$\Sigma(z) = -(z/6 + z^2/144 + z^3/2880 + 7z^4/414720 + \dots) \quad z_0 = \arcsin(\sqrt{3/8})$$

All loop Exact Recurrence Relation

D=8 N=1

s-channel term $S_n(s, t)$ **t-channel term** $T_n(s, t)$ $T_n(s, t) = S_n(t, s)$

Exact relation for ALL diagrams

$$\begin{aligned} nS_n(s, t) &= -2s^2 \int_0^1 dx \int_0^x dy \ y(1-x) \ (S_{n-1}(s, t') + T_{n-1}(s, t'))|_{t'=tx+yu} \\ &+ s^4 \int_0^1 dx \ x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!} \ \frac{d^p}{dt'^p} (S_k(s, t') + T_k(s, t')) \times \\ &S_1 = \frac{1}{12}, \ T_1 = \frac{1}{12} \quad \times \frac{d^p}{dt'^p} (S_{n-1-k}(s, t') + T_{n-1-k}(s, t'))|_{t'=-sx} \ (tsx(1-x))^p \end{aligned}$$

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D=8 N=1

s-channel term $S_n(s, t)$ **t-channel term** $T_n(s, t)$ $T_n(s, t) = S_n(t, s)$

Exact relation for ALL diagrams

$$\begin{aligned} nS_n(s, t) &= -2s^2 \int_0^1 dx \int_0^x dy \ y(1-x) \ (S_{n-1}(s, t') + T_{n-1}(s, t'))|_{t'=tx+yu} \\ &+ s^4 \int_0^1 dx \ x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!} \ \frac{d^p}{dt'^p} (S_k(s, t') + T_k(s, t')) \times \\ &S_1 = \frac{1}{12}, \ T_1 = \frac{1}{12} \quad \times \frac{d^p}{dt'^p} (S_{n-1-k}(s, t') + T_{n-1-k}(s, t'))|_{t'=-sx} \ (tsx(1-x))^p \end{aligned}$$

summation

All loop Exact Recurrence Relation

D=8 N=1

s-channel term $S_n(s, t)$ **t-channel term** $T_n(s, t)$ $T_n(s, t) = S_n(t, s)$

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summation $\Sigma_3(s, t, z) = \Sigma_1(s, t, z) - S_2(s, t)z^2 + S_1(s, t)z, \ \Sigma_2(s, t, z) = \Sigma_1(s, t, z) + S_1(s, t)z$

All loop Exact Recurrence Relation

D=8 N=1

s-channel term	$S_n(s, t)$	t-channel term	$T_n(s, t)$
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$$T_n(s, t) = S_n(t, s)$$

Exact relation for ALL diagrams

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$$+ s^4 \int_0^1 dx \ x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!} \ \frac{d^p}{dt'^p} (S_k(s, t') + T_k(s, t')) \times$$

$$S_1 = \frac{1}{12}, \ T_1 = \frac{1}{12} \quad \times \frac{d^p}{dt'^p} (S_{n-1-k}(s, t') + T_{n-1-k}(s, t'))|_{t'=-sx} \ (tsx(1-x))^p$$

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Diff eqn

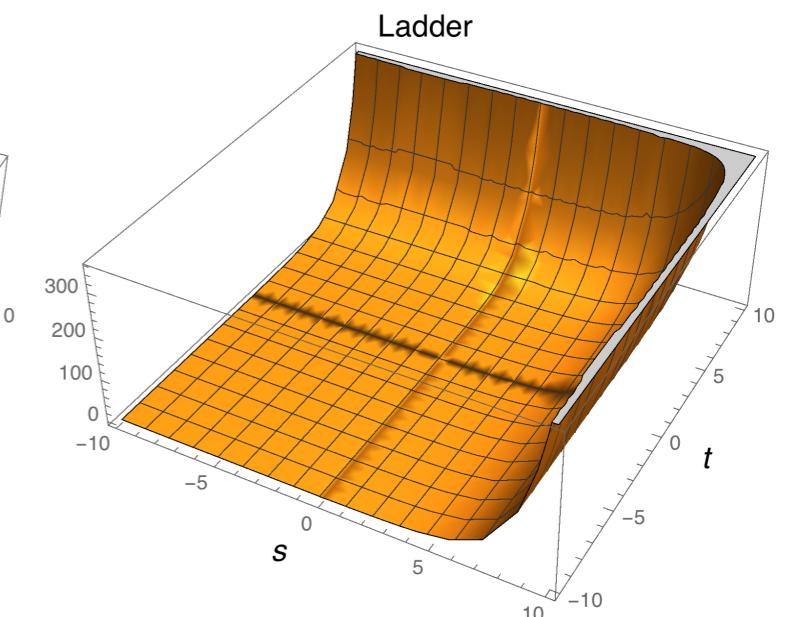
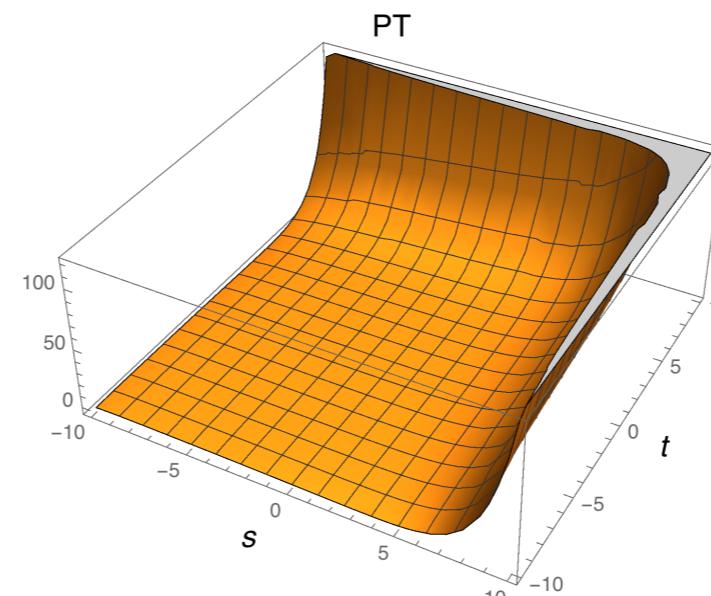
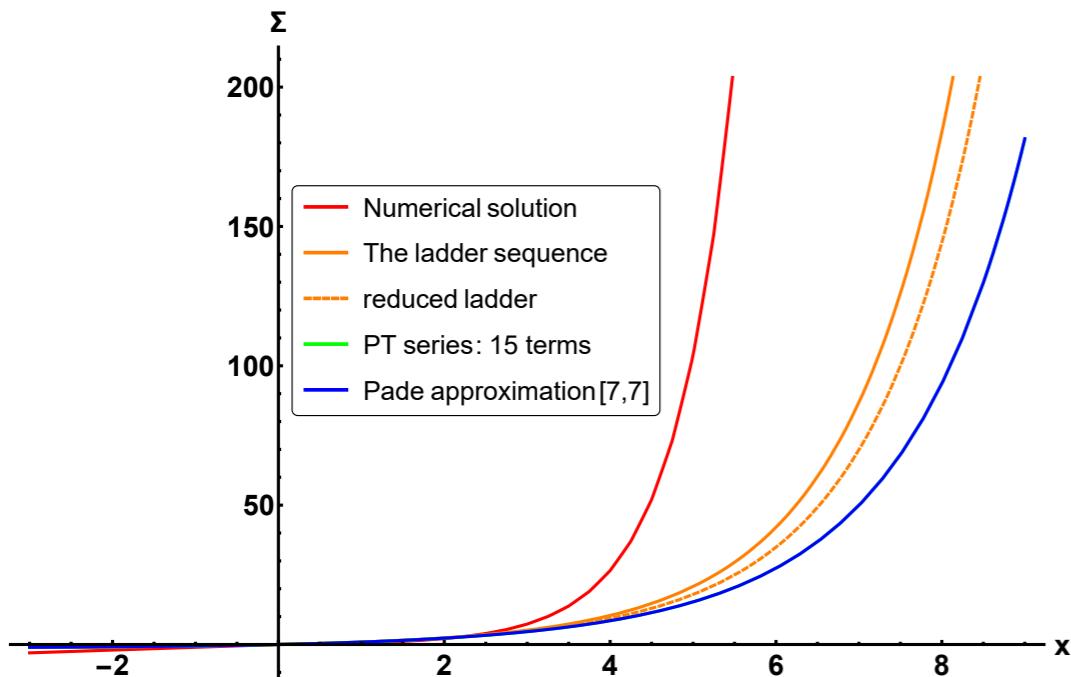
$$\frac{d}{dz} \Sigma(s, t, z) = -\frac{1}{12} + 2s^2 \int_0^1 dx \int_0^x dy \ y(1-x) \ (\Sigma(s, t', z) + \Sigma(t', s, z))|_{t'=tx+yu}$$

$$-s^4 \int_0^1 dx \ x^2(1-x)^2 \sum_{p=0}^{\infty} \frac{1}{p!(p+2)!} \left(\frac{d^p}{dt'^p} (\Sigma(s, t', z) + \Sigma(t', s, z))|_{t'=-sx} \right)^2 (tsx(1-x))^p.$$

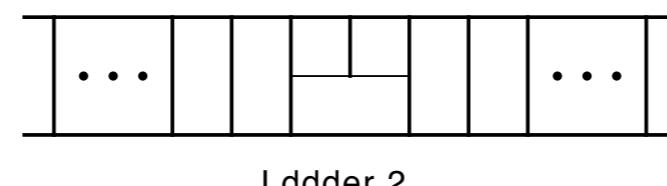
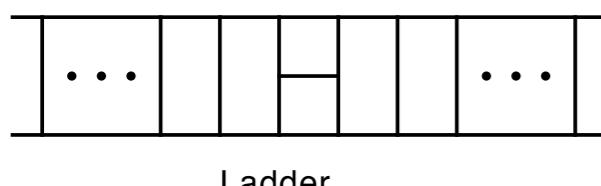
All loop Solution (leading divs)

D=6 N=2

PT (15 terms)

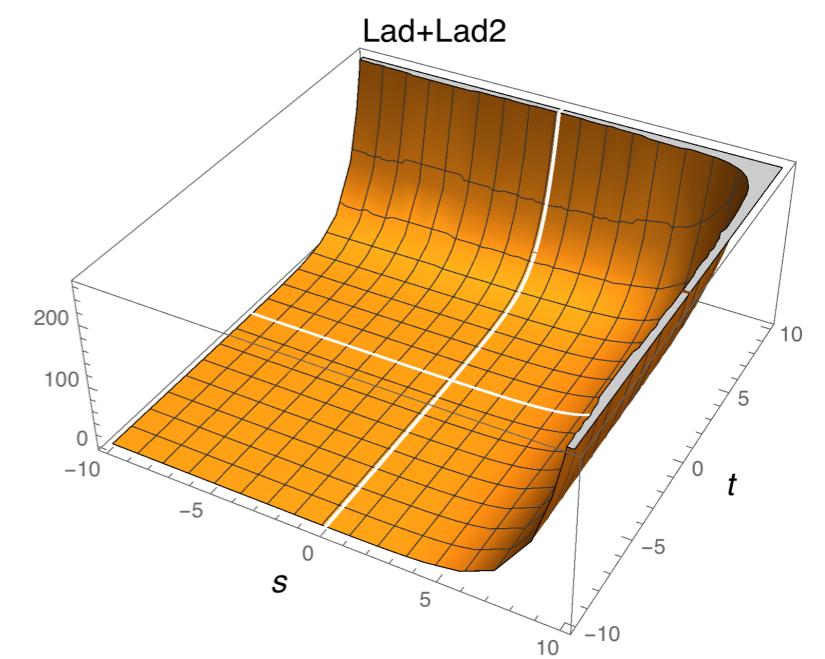


PT and Pade versus
ladder for $t=s$



$$\Sigma_L(s, z) = \frac{2}{s^2 z^2} \left(e^{sz} - 1 - sz - \frac{s^2 z^2}{2} \right)$$

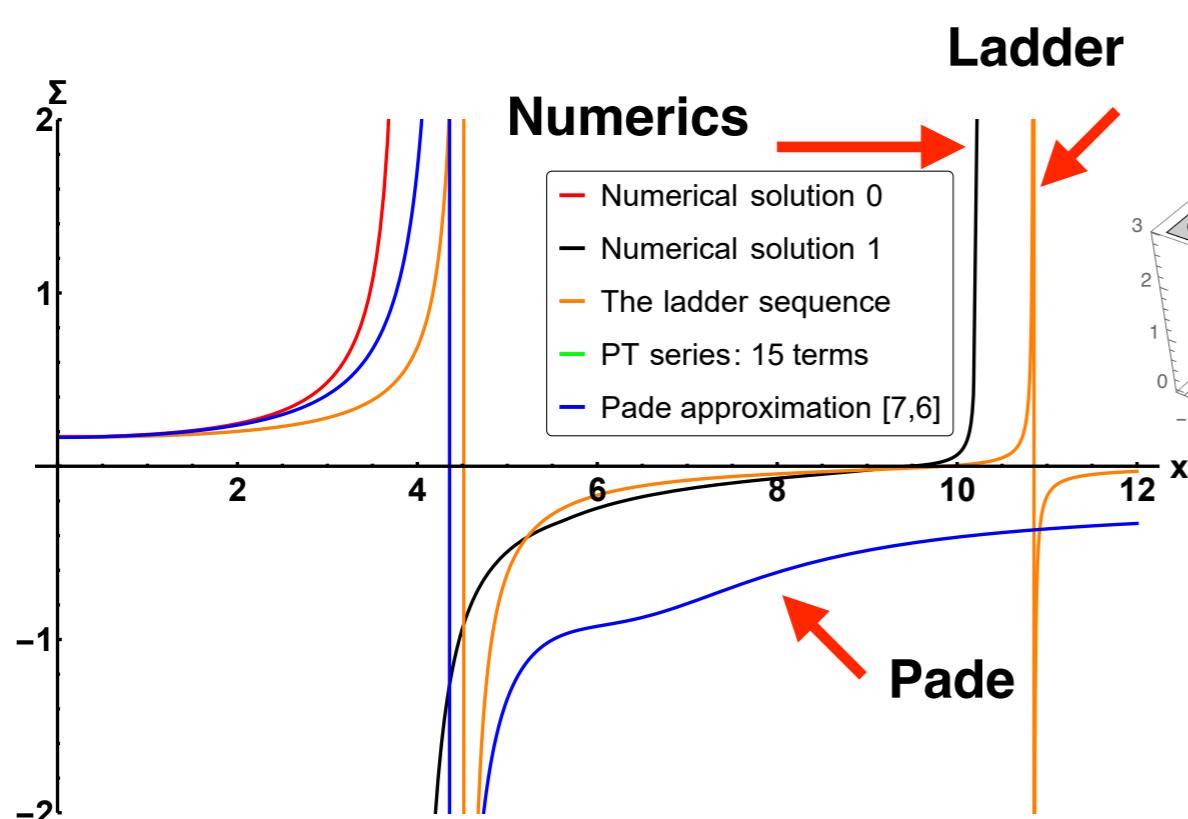
$$z = \frac{g^2}{\epsilon}$$



Numerical solution of the full equation is close to the ladder approx

All loop Solution (leading divs)

D=8 N=1

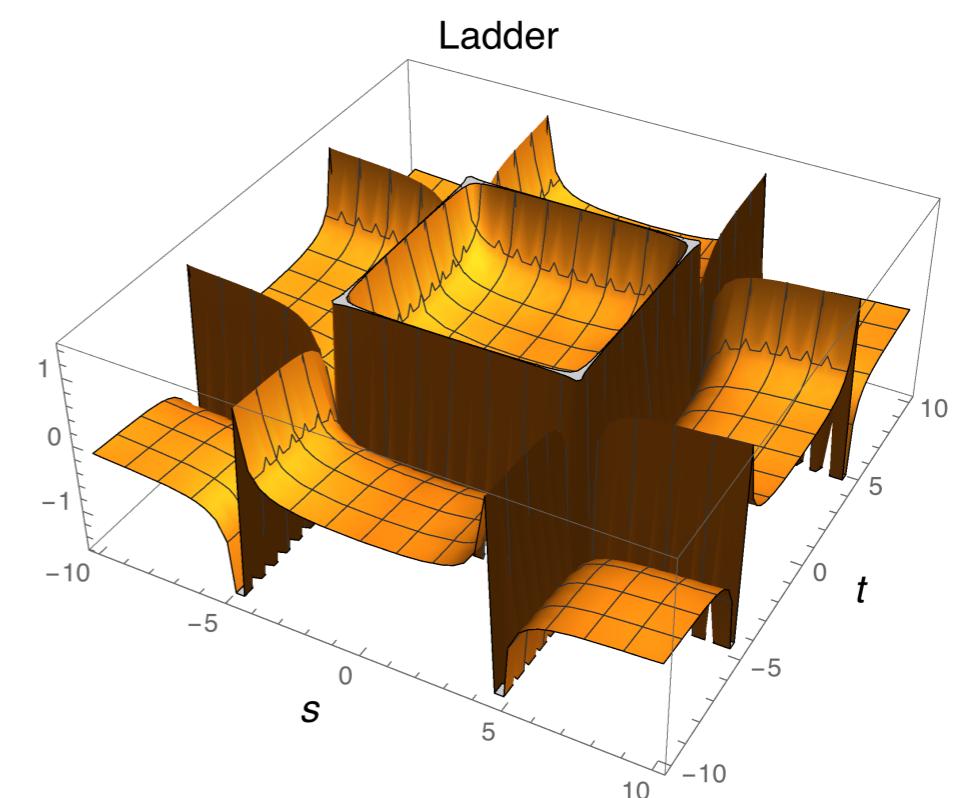
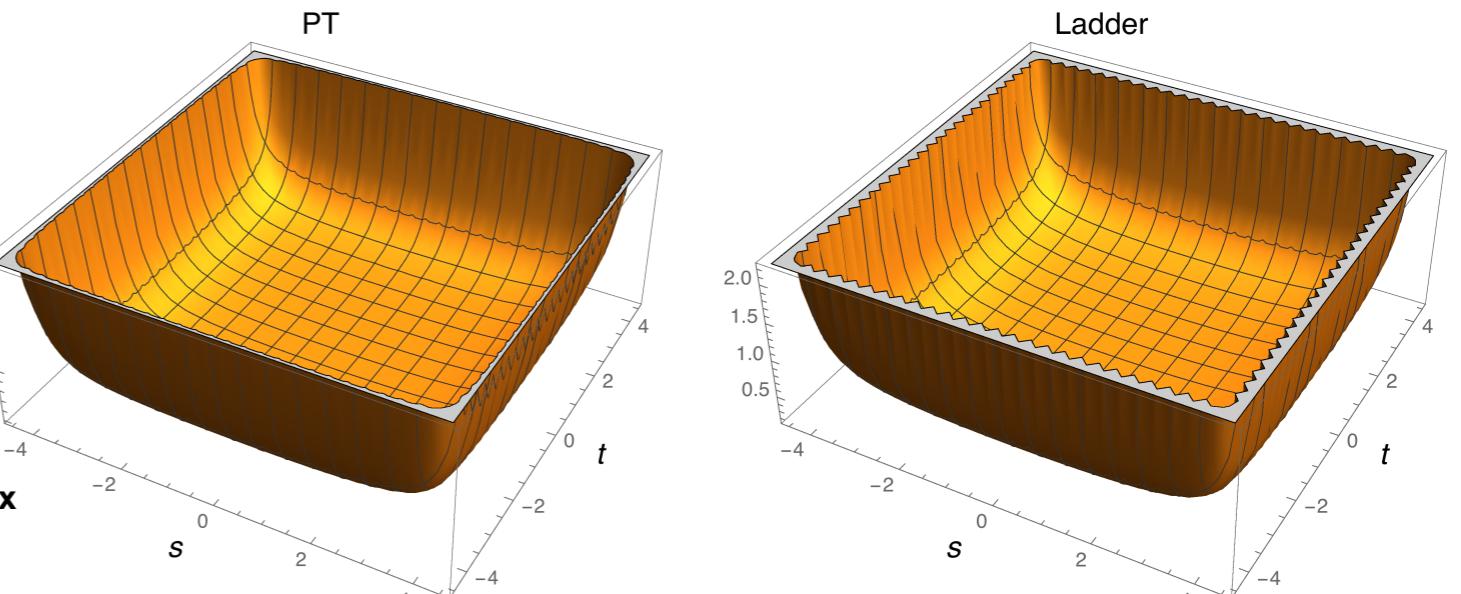


PT and Pade versus
ladder for $t=s$



$$z = \frac{g^2}{\epsilon}$$

$$\Sigma_L(s, z) = -\sqrt{5/3} \frac{4 \tan(zs^2/(8\sqrt{15}))}{1 - \tan(zs^2/(8\sqrt{15}))\sqrt{5/3}}$$



Subleading divergences

$$\Sigma_L(z) + \epsilon \Sigma_{NL}(z) + \epsilon^2 \Sigma_{NNL}(z) + \dots \quad \Sigma(z) = \sum_n^\infty z^n F_n$$

$$D = 4 \quad N = 4 \quad z = g^2/\epsilon$$

$$D = 6 \quad N = 2 \quad z = g^2 s/\epsilon, z = g^2 t/\epsilon$$

$$D = 8 \quad N = 1 \quad z = g^2 s^2/\epsilon, z = g^2 s t/\epsilon, ..$$

$$D = 10 \quad N = 1 \quad z = g^2 s^3/\epsilon, z = g^2 s^2 t/\epsilon, ..$$

D=8 N=1

sLadder case

$$\Sigma_{NL} = s \Sigma_{sB}(z) + t \Sigma_{tB}(z)$$

$$z = \frac{g^2 s^2}{\epsilon}$$

Subleading divergences

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D=8 N=1

sLadder case $\Sigma_{NL} = s \Sigma_{sB}(z) + t \Sigma_{tB}(z)$ $z = \frac{g^2 s^2}{\epsilon}$

$$\Sigma'_{tB}(z) = \frac{5}{6} \left[e^{z/60} (2 \cos(z/30) - \sin(z/30)) - 2 \right]$$

$$\Sigma_{tB} = -\frac{1}{36} \left[60 + z + e^{z/60} (-(60 + z) \cos(z/30) - 2(-15 + z) \sin(z/30)) \right]$$

Sum of Ladder diagrams (subleading divs)

$$\Sigma'_{sB} = \sum_{n=2}^{\infty} z^n B'_{sn}$$

$$\frac{d^2 \Sigma'_{sB}(z)}{dz^2} + f_1(z) \frac{d \Sigma'_{sB}(z)}{dz} + f_2(z) \Sigma'_{sB}(z) = f_3(z)$$

Diff eqn

$$f_1(z) = -\frac{1}{6} + \frac{\Sigma_A}{15},$$

$$f_2(z) = \frac{1}{80} - \frac{\Sigma_A}{360} + \frac{\Sigma_A^2}{600} + \frac{1}{15} \frac{d\Sigma_A}{dz},$$

$$\begin{aligned} f_3(z) = & \frac{2321}{5!5!2} \Sigma_A + \frac{11}{1800} \Sigma'_{tB} - \frac{47}{5!45} \Sigma_A^2 - \frac{1}{5!72} \Sigma_A \Sigma'_{tB} + \frac{23}{6750} \Sigma_A^3 + \frac{1}{1200} \Sigma_A^2 \Sigma'_{tB} \\ & - \frac{19}{36} \frac{d\Sigma_A}{dz} - \frac{1}{15} \frac{d\Sigma'_{tB}}{dz} + \frac{23}{225} \frac{d\Sigma_A^2}{dz} + \frac{1}{30} \frac{d(\Sigma_A \Sigma'_{tB})}{dz} - \frac{3}{32} \end{aligned}$$

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Solution to Diff eqn

$$\Sigma'_{sB}(z) = \frac{d \Sigma_A}{dz} u(z)$$

$$u(z) = \int_0^z dy \int_0^y dx \frac{f_3(x)}{d \Sigma_A(x)/dx}$$

$$f_1(z) = -\frac{1}{6} + \frac{\Sigma_A}{15},$$

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smooth monotonic function



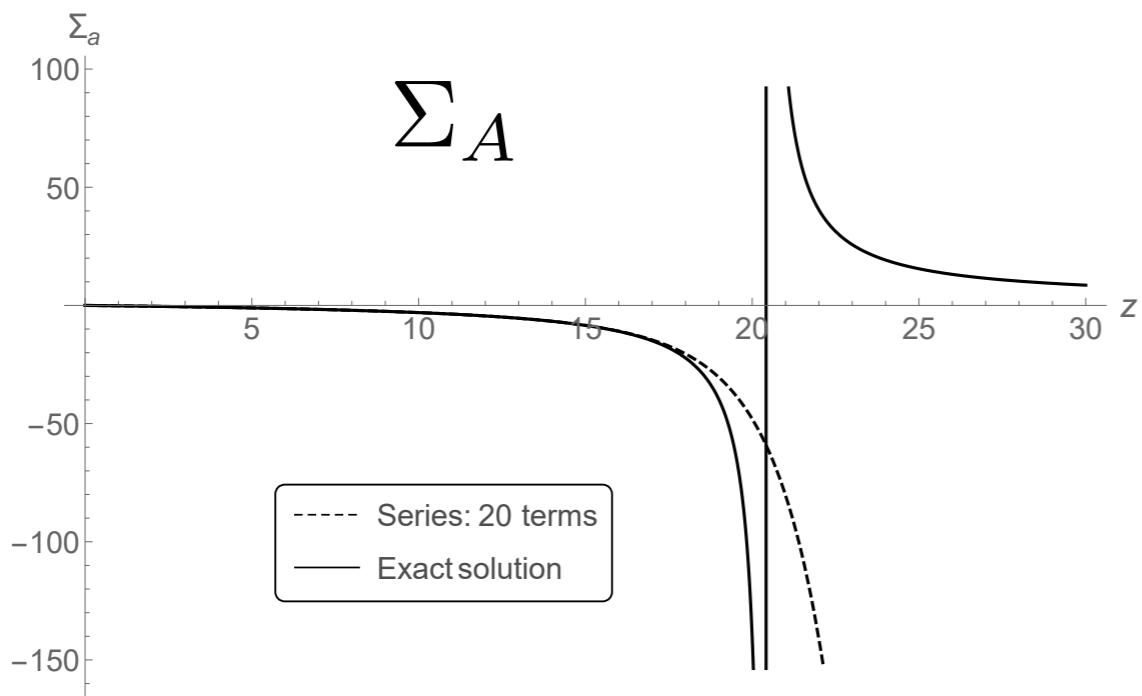
Sum of the Ladder diagrams

solutions

Sum of the Ladder diagrams

solutions

Leading divs

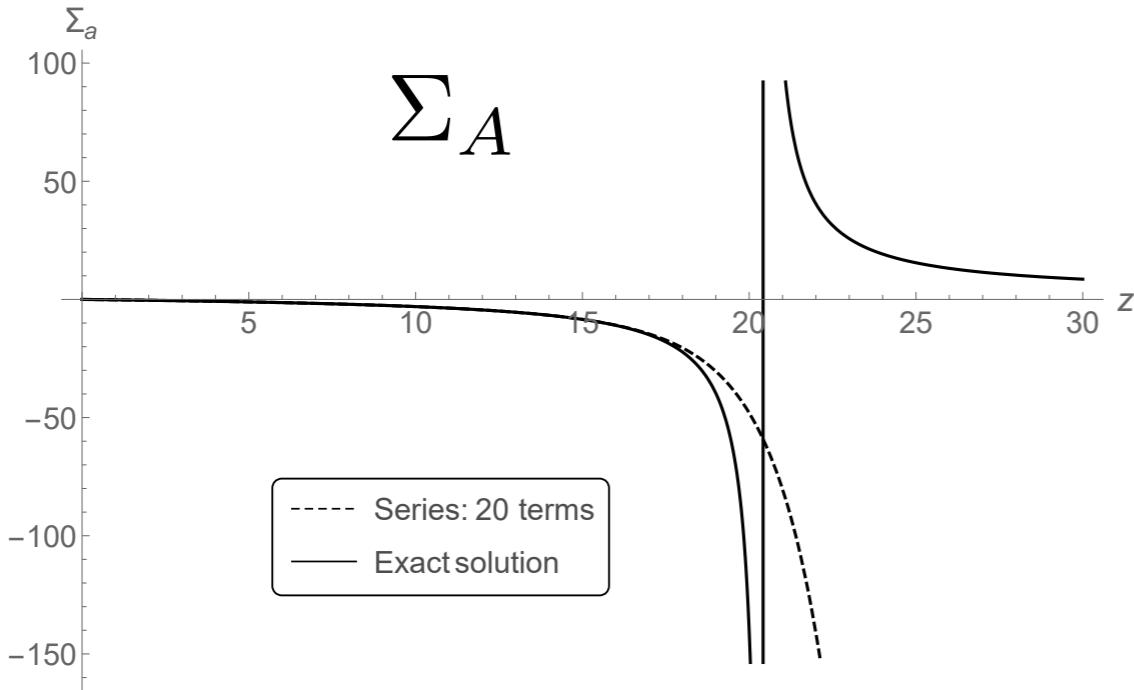


Infinite number of poles

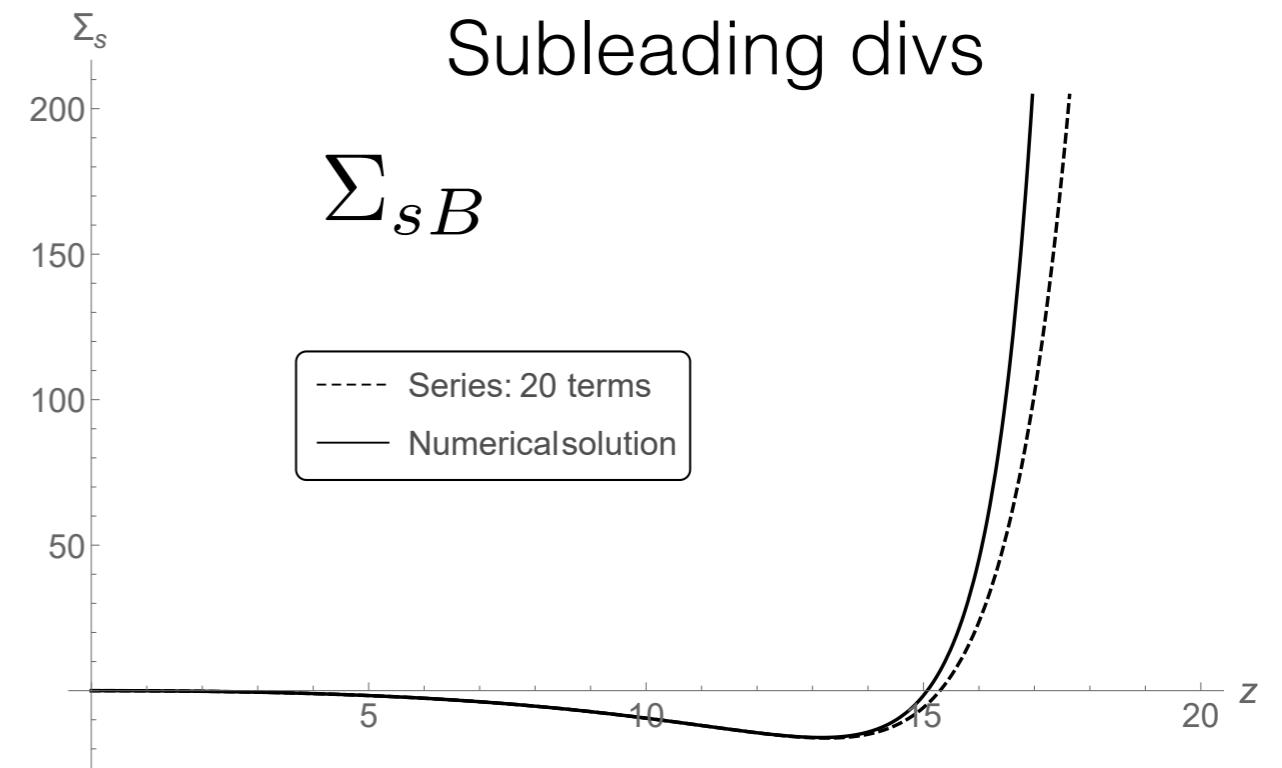
Sum of the Ladder diagrams

solutions

Leading divs



Subleading divs

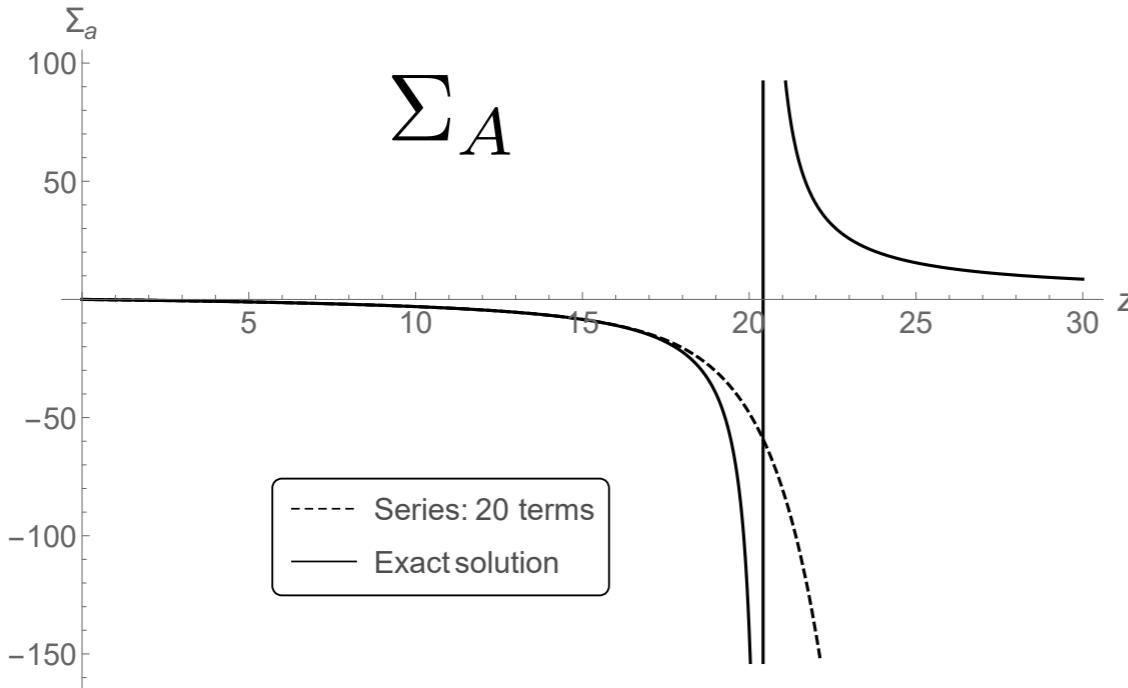


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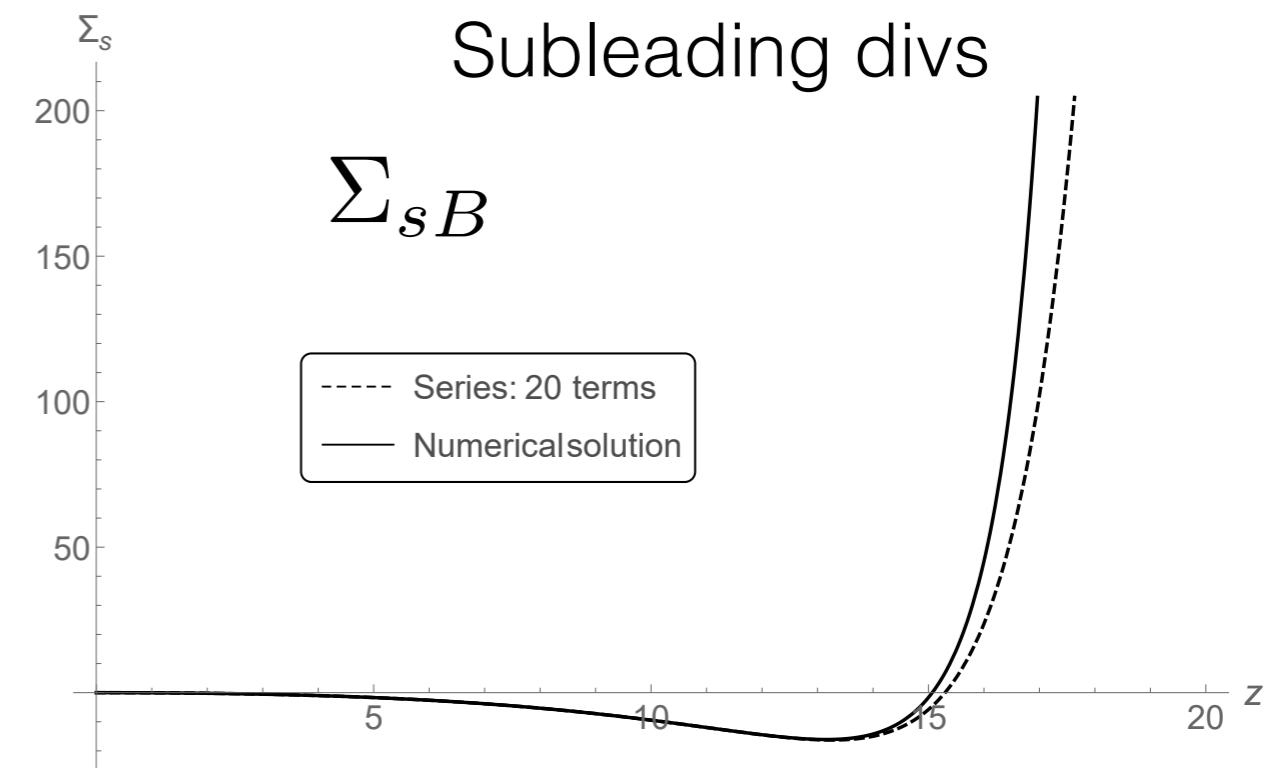
Sum of the Ladder diagrams

solutions

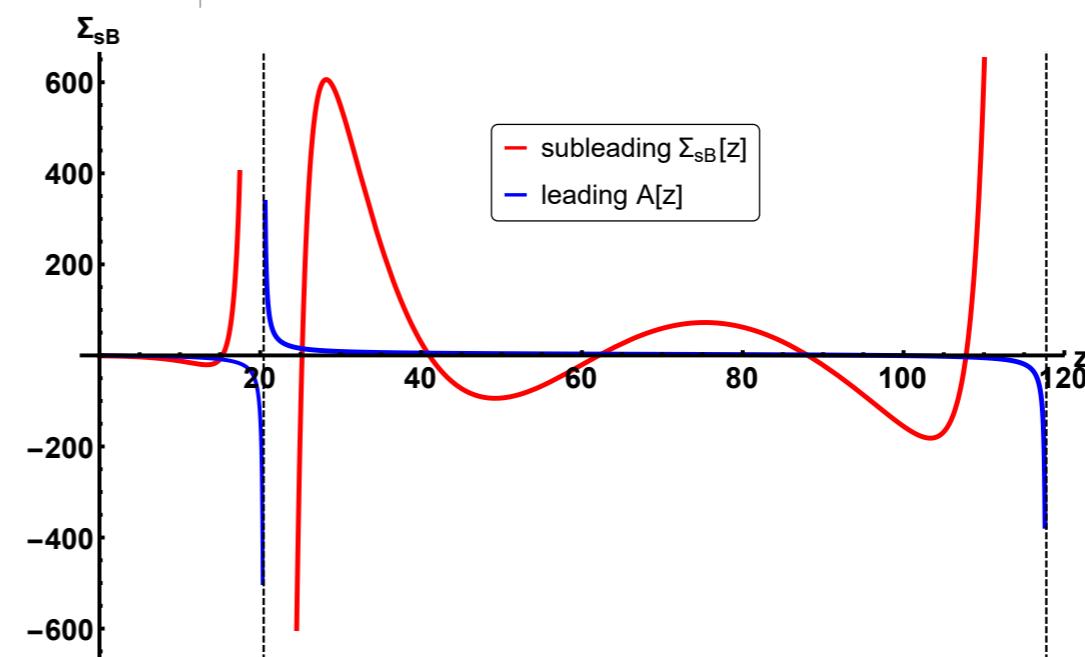
Leading divs



Subleading divs



Infinite number of poles



Infinite number of poles at the same position

Scheme dependence and arbitrariness of subtraction

subleading case

$$A'_1 + B'_{s1} = \frac{1}{6\epsilon}(1 + c_1\epsilon) \quad \Delta\Sigma'_{sB} = c_1 z \frac{d\Sigma'_A}{dz}. \quad \rightarrow \quad z \rightarrow z(1 + c_1\epsilon).$$

sub-subleading case

$$A'_2 + B'_2 = \frac{s}{3!4!\epsilon^2} \left(1 - \frac{5}{12}\epsilon + 2c_1\epsilon + c_2\epsilon^2 \right) \quad \Delta\Sigma'_{sC} = c_2 z^2 \frac{d\Sigma'_A}{dz}.$$

$$\rightarrow \quad z \rightarrow z(1 + c_1\epsilon) + z^2 c_2 \epsilon^2.$$

$$\Delta\Sigma'_{sC} = -c_1^2 \frac{z}{4!} \left(\frac{d\Sigma'_A}{dz} - 12 \frac{d^2\Sigma'_A}{dz^2} \right) \quad \rightarrow \quad z \rightarrow z(1 + c_1\epsilon) + z^2(c_2 + c_1^2/4!) \epsilon^2$$

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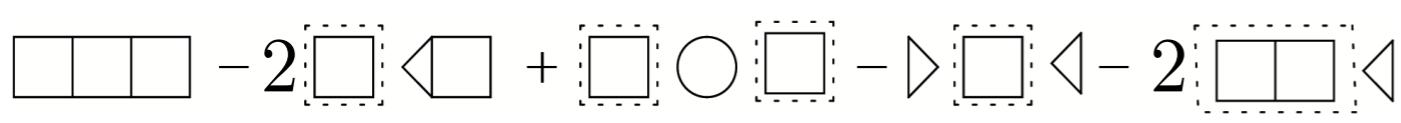
linear term

new contribution from
subleading term

$$\Delta\Sigma'_{sC}(3-loop) = -\frac{719c_1s^2}{1036800\epsilon}$$



$$\Sigma'_{sB}(3-loop) = -\frac{71s^2}{345600\epsilon^2}$$



$$\Sigma'^{trunc}_{sB}(3-loop) = -\frac{719s^2}{3110400\epsilon^2}$$

the source of
a problem

$$\Delta\Sigma'_{sC}(3-loop) = c_1 z \frac{d\Sigma'^{trunc}_{sB}}{dz}(3-loop)$$

$$z \rightarrow z(1 + c_1\epsilon) + z^2(c_2 - c_1^2/4!) \epsilon^2 + z^3 c_1^3/6! \epsilon^3 - z^4 c_1^4/4!6! \epsilon^4 + \dots$$

Scheme dependence and arbitrariness of subtraction

sub-subleading case

$$A'_2 + B'_2 = \frac{s}{3!4!\epsilon^2} \left(1 - \frac{5}{12}\epsilon + 2c_1\epsilon + c_2\epsilon^2 \right)$$

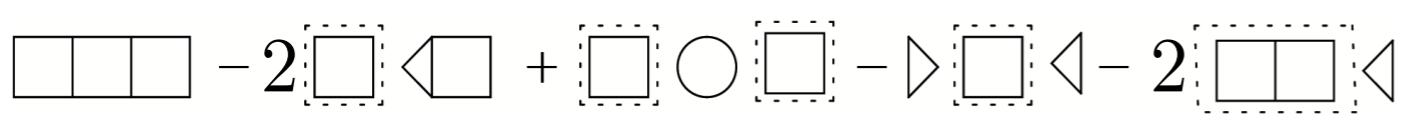
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Kinematically dependent renormalization

- R-operation is equivalent to

renormalizable theories

nonrenormalizable theories

$$\bar{A}_4 = Z_4(g^2) \bar{A}_4^{bare} \Big|_{g_{bare}^2 \rightarrow g^2} Z_4$$

$$g_{bare}^2 = \mu^\epsilon Z_4(g^2) g^2.$$

$$Z = 1 - \sum_i K R' G_i$$

simple multiplication

operator multiplication

$$Z = 1 + \frac{g^2}{\epsilon} + g^4 \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \right) + \dots$$

scheme dependence

$$g^2 = z g'^2, \quad z = 1 + g'^2 c_1 + g'^4 c_2 + \dots$$

$$Z = 1 + \frac{g^2}{\epsilon} st + g'^4 st \left(\frac{s^2 + t^2}{\epsilon^2} + \frac{s^2 + st + t^2}{\epsilon} \right) + \dots$$

scheme dependence

$$g^2 = z g'^2, \quad z = 1 + g'^2 st c_1 + g'^4 st (s^2 + t^2) c_2 + \dots$$

Kinematically dependent renormalization

operator kinematically dependent renormalization

at 2 loops

$$\bar{A}_4 = 1 - \frac{g_B^2 st}{3! \epsilon} - \frac{g_B^4 st}{3! 4!} \left(\frac{s^2 + t^2}{\epsilon^2} + \frac{27/4 s^2 + 1/3 st + 27/4 t^2}{\epsilon} \right) + \dots$$

$$\bar{A}_4 = Z_4(g^2) \bar{A}_4^{bare} |_{g_{bare}^2 \rightarrow g^2 Z_4}$$

$$Z_4 = 1 + \frac{g^2 st}{3! \epsilon} + \frac{g^4 st}{3! 4!} \left(-\frac{s^2 + t^2}{\epsilon^2} + \frac{5/12 s^2 + 1/3 st + 5/12 t^2}{\epsilon} \right)$$

$$g_B^2 = g^2 \left(1 + \frac{g^2}{3! \epsilon} \right)$$

$$g^2 s t \quad \boxed{} \quad \Rightarrow \quad g^2 \left(s \quad \triangle \quad + \quad t \quad \nabla \right)$$

this is operator action!

Conclusions

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- ➋ The recurrence relations allow one to calculate the leading UV divergences in ALL orders of PT algebraically starting from 1 loop
- ➌ The recurrence relations allow one to calculate the sub leading UV divergences in ALL orders of PT algebraically starting from 1 and 2 loops
- ➍ This procedure apparently continues the same way for all divergences just like in renormalizable theories

Conclusions cont'd

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- The sum of the leading UV divergences to ALL orders obeys the nonlinear integro-differential equation

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- The sum of the leading UV divergences to ALL orders obeys the nonlinear integro-differential equation
- The numerical solution indicates that solution to the full equation seems to behave like the ladder approximation
- There is no simple limit when $\epsilon \rightarrow +0$
- This means that one cannot simply remove the UV divergence and non-renormalizability of a theory is not improved when summing the infinite series

Conclusions cont'd

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- 💡 **The structure of UV divergences in non-renormalizable theories essentially copies that of renormalizable ones**

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- The structure of UV divergences in non-renormalizable theories essentially copies that of renormalizable ones
- The main difference is that the renormalization constant depends on kinematics and acts like an operator rather than simple multiplication
- As a result, one can construct the higher derivative theory that gives the finite scattering amplitudes with a single arbitrary coupling g defined in PT within the given renormalization scheme.
- Transition to another scheme is performed by the action on the amplitude of a finite renormalization operator z that depends on kinematics.