QUANTUM SPECTRAL CURVE AND CORRELATORS IN N=4 SYM

Fedor Levkovich-Maslyuk

Ecole Normale Superieure Paris
and IITP Moscow

based on 1802.04237

with Andrea Cavaglia and Nikolay Gromov

N=4 SUPER YANG-MILLS

Highly nontrivial CFT in 4d

$$S = \frac{1}{g_{YM}^2} \int d^4x \; \mathrm{tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi_i)^2 - \frac{1}{2} [\Phi_i, \Phi_j]^2 + \text{fermions} \right\}$$

At large N_c exact solution may be possible due to integrability

$N_c \to \infty$, $\lambda = g_{YM}^2 N_c$ is fixed – 't Hooft coupling

Some motivation:

- Solvable gauge theory in 4d
- Directly related to QCD in some limits (BFKL) — Balitsky, Fadin, Kuraev, Lipatov
- Understanding AdS/CFT duality

MOTIVATION

CFT $\Rightarrow$ study correlators

$$\mathcal{O}(x) = \mathrm{Tr}\,(\Phi_1\Phi_2\Phi_3\ldots)\,(x)$$

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|^{2\Delta}}$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}$$

$\Delta_i(\lambda)$ and $C_{ijk}(\lambda)$ are key observables

spectrum

structure constants

MOTIVATION

Quantum Spectral Curve (QSC) – very powerful method to compute the spectrum in planar N=4 SYM

Based on integrability

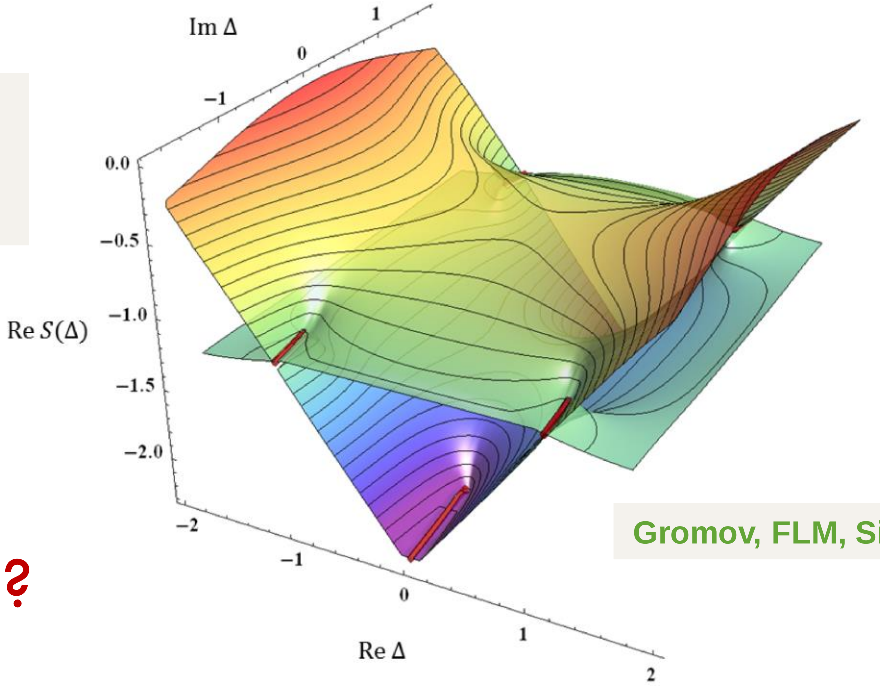
Gromov, Kazakov, Leurent, Volin 2013

10+ loops at weak coupling, numerics with 60-digit precision, BFKL, …

Marboe, Volin 14-17
Alfimov, Gromov, Kazakov 14,18
Gromov, FLM, Sizov 14, 15
…

Also nonlocal operators, q-q potential

Gromov, FLM 15, 16



Gromov, FLM, Sizov 15

Is there an analog for 3-pt correlators ?

QUANTUM SPECTRAL CURVE (QSC)

See 1708.03648 [Gromov] for an introduction

INSPIRING EXAMPLE

Harmonic oscillator:

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$

$$\psi(x) = e^{-\frac{m\omega x^2}{2\hbar}} Q(x) \quad , \quad Q(x) \equiv \prod_{i=1}^{N}(x - x_i) \qquad \psi_2 \simeq x^{-N-1} e^{+\frac{m\omega}{2h}x^2}$$

$$W = \begin{vmatrix} \psi_1(x) & \psi_1'(x) \\ \psi_2(x) & \psi_2'(x) \end{vmatrix}$$ is a constant

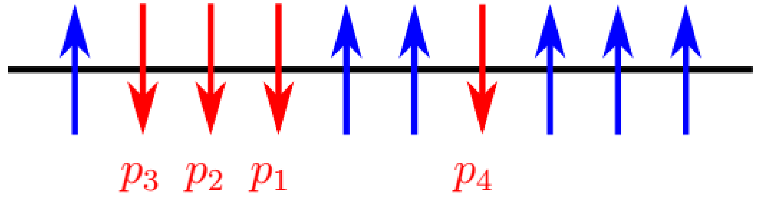
XXX SPIN CHAINS

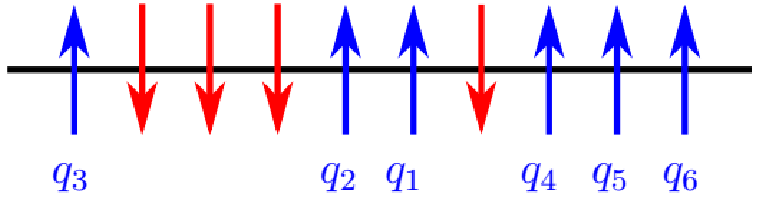
Starting point: Baxter equation

$$T(u)Q(u) + (u+i/2)^L Q(u-i) + (u-i/2)^L Q(u+i) = 0$$

Two solutions: polynomials $\quad Q_1 \sim u^N \qquad Q_1 = \prod (u - u_i)$

second solution (for the same T) $\quad Q_2 \sim u^{L-N}$





L-spins, N-spins up, L-N down

Equivalent description:

$$\begin{vmatrix} Q_1\left(u+\frac{i}{2}\right) & Q_2\left(u+\frac{i}{2}\right) \\ Q_1\left(u-\frac{i}{2}\right) & Q_2\left(u-\frac{i}{2}\right) \end{vmatrix} = u^L$$

Easy to generalize to SU(3):

$$\begin{vmatrix} Q_1(u+i) & Q_2(u+i) & Q_3(u+i) \\ Q_1(u) & Q_2(u) & Q_3(u) \\ Q_1(u-i) & Q_2(u-i) & Q_3(u-i) \end{vmatrix} = u^L$$

GENERALIZATION TO N=4 SYM

Two main ingredients:

$$\begin{vmatrix} Q_1\left(u+\frac{i}{2}\right) & Q_2\left(u+\frac{i}{2}\right) \\ Q_1\left(u-\frac{i}{2}\right) & Q_2\left(u-\frac{i}{2}\right) \end{vmatrix} = u^L$$

- QQ-relations

$$su(2) \to psu(2,2|4)$$

$$(Q_1,\ Q_2) \quad \longrightarrow \quad (\underbrace{\mathbf{P}_1,\ \mathbf{P}_2,\ \mathbf{P}_3,\ \mathbf{P}_4}_{S^5}|\underbrace{\mathbf{Q}_1,\ \mathbf{Q}_2,\ \mathbf{Q}_3,\ \mathbf{Q}_4}_{AdS_5})$$

- Analyticity

Q_1 - polynomial

Q_2 - polynomial

In N=4 SYM polynomials are replaced by analytic functions with cuts and monodromy condition

QQ-relations + monodromy = Quantum Spectral Curve

Gromov, Kazakov, Leurent, Volin 2013

BAXTER EQUATION

$$\mathbf{Q}_i^{[+4]} D_0 - \mathbf{Q}_i^{[+2]} \left[D_1 - \mathbf{P}_a^{[+2]} \mathbf{P}^{a[+4]} D_0 \right] + \mathbf{Q}_i \left[D_2 - \mathbf{P}_a \mathbf{P}^{a[+2]} D_1 + \mathbf{P}_a \mathbf{P}^{a[+4]} D_0 \right]$$
$$- \mathbf{Q}_i^{[-2]} \left[\bar{D}_1 + \mathbf{P}_a^{[-2]} \mathbf{P}^{a[-4]} \bar{D}_0 \right] + \mathbf{Q}_i^{[-4]} \bar{D}_0 = 0$$

$$D_0 = \det \begin{pmatrix} \mathbf{P}^{1[+2]} & \mathbf{P}^{2[+2]} & \mathbf{P}^{3[+2]} & \mathbf{P}^{4[+2]} \\ \mathbf{P}^{1} & \mathbf{P}^{2} & \mathbf{P}^{3} & \mathbf{P}^{4} \\ \mathbf{P}^{1[-2]} & \mathbf{P}^{2[-2]} & \mathbf{P}^{3[-2]} & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \mathbf{P}^{2[-4]} & \mathbf{P}^{3[-4]} & \mathbf{P}^{4[-4]} \end{pmatrix} \qquad D_1 = \det \begin{pmatrix} \mathbf{P}^{1[+4]} & \mathbf{P}^{2[+4]} & \mathbf{P}^{3[+4]} & \mathbf{P}^{4[+4]} \\ \mathbf{P}^{1} & \mathbf{P}^{2} & \mathbf{P}^{3} & \mathbf{P}^{4} \\ \mathbf{P}^{1[-2]} & \mathbf{P}^{2[-2]} & \mathbf{P}^{3[-2]} & \mathbf{P}^{4[-2]} \\ \mathbf{P}^{1[-4]} & \mathbf{P}^{2[-4]} & \mathbf{P}^{3[-4]} & \mathbf{P}^{4[-4]} \end{pmatrix}$$

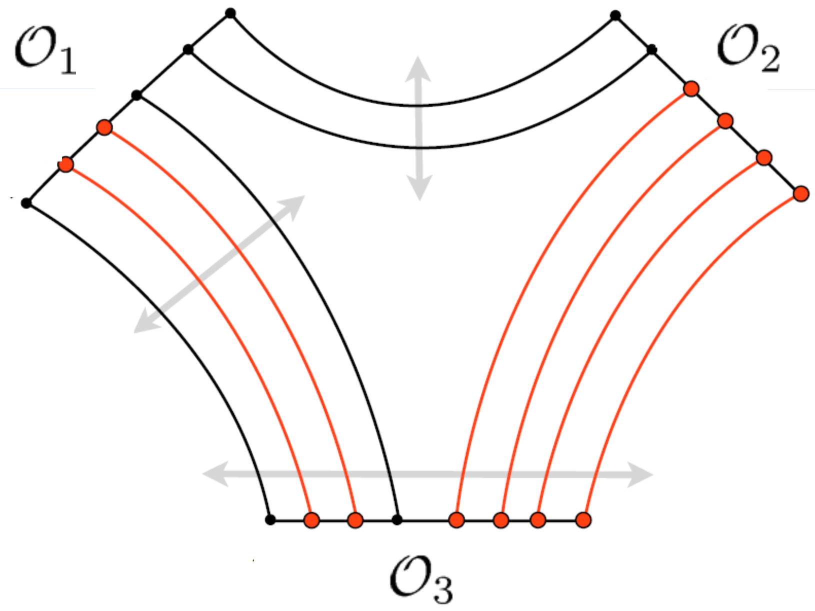
Solve this equations => spectrum of anomalous dimensions of all local (and not only) operators

Sklyanin 92

The Q-functions should correspond to wavefunction in separated variables

$$\Psi \sim Q(x_1)Q(x_2)\dots Q(x_n)$$

The 3pt correlators should be some scalar product of 3 Q-functions



Indeed we will see this explicitly!

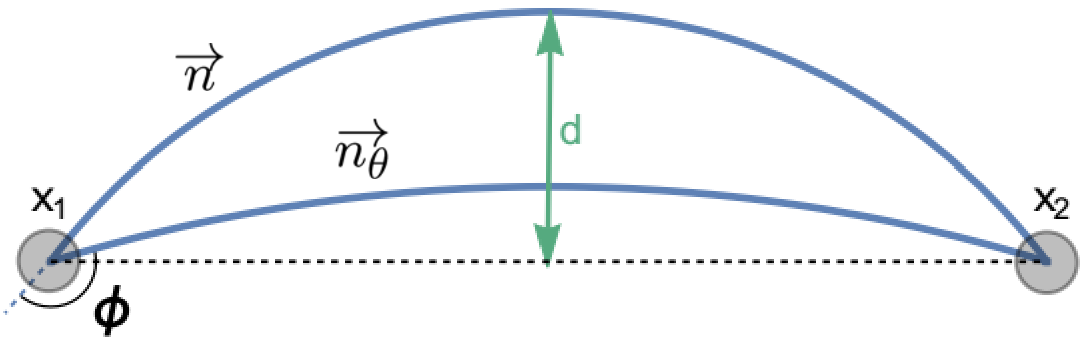
Instead of local operators we look at Wilson lines, can compute an all-orders 3pt correlator

SET-UP: WHAT WE ARE GOING TO COMPUTE

CUSPED WILSON LINE IN N=4 SYM

Similar to a 2pt function

$$\langle W \rangle \sim \left(\frac{\Lambda_{IR}}{\Lambda_{UV}} \right)^{\Delta}$$



$$W = \mathrm{Tr}\, \mathcal{P} \exp \int dt \left[iA \cdot \dot{x} + \vec{\Phi} \cdot \vec{n}\, |\dot{x}| \right]$$

Drukker,Forini 11
Drukker 12
Correa,Maldacena,Sever 12

Parameters:

- Cusp angle ϕ
- Angle θ between the couplings to scalars on two rays $\vec{n} \cdot \vec{n}_\theta = \cos\theta$
- ‘t Hooft coupling λ

Described by the same QSC !

Gromov, FLM 15

7-loop result. The term of order $\hat{g}^{14}$ in $\frac{\Omega}{4\pi}$ is given by

Gromov, FLM 2016

$$
\begin{aligned}
&\frac{1048576L^6}{45} + \frac{524288}{9}L^5\pi^2T + \frac{6815744L^5}{15} + \frac{262144}{9}L^4\pi^4T^2 - 65536L^4T\zeta_3 + \frac{40632320}{9}L^4\pi^2T \\
&-\frac{15007744}{9}L^4\pi^2 + 2752512L^4 + \frac{131072}{81}L^3\pi^6T^3 + 65536L^3\pi^2T^2\zeta_3 + \frac{655360}{3}L^3T^2\zeta_5 \\
&+\frac{12255232}{9}L^3\pi^4T^2 - \frac{64159744}{135}L^3\pi^4T - 65536L^3T\zeta_3 + \frac{13303808}{3}L^3\pi^2T + \frac{3407872L^3\zeta_3}{9} \\
&-\frac{11141120}{9}L^3\pi^2 + \frac{15073280L^3}{3} + \frac{2080768}{45}L^2\pi^4T^3\zeta_3 - \frac{499712}{3}L^2\pi^2T^3\zeta_5 - 129024L^2T^3\zeta_7 \\
&+32768L^2\pi^6T^3 - \frac{2828288}{405}L^2\pi^6T^2 - 36864L^2T^2\zeta_3^2 + \frac{11444224}{3}L^2\pi^2T^2\zeta_3 + 20480L^2T^2\zeta_5 \\
&+\frac{2351104}{3}L^2\pi^4T^2 - \frac{7610368}{9}L^2\pi^2T\zeta_3 - 40960L^2T\zeta_5 - \frac{27344896}{45}L^2\pi^4T + 1671168L^2T\zeta_3 \\
&-3817472L^2\pi^2T + \frac{7221248L^2\pi^4}{45} + 2555904L^2\zeta_3 + \frac{17096704L^2\pi^2}{9} - \frac{6914048L^2}{3} + \frac{8192}{9}L\pi^6T^4\zeta_3 \\
&-\frac{133120}{3}L\pi^4T^4\zeta_5 + 369152L\pi^2T^4\zeta_7 - 628992LT^4\zeta_9 + \frac{1176832L\pi^8T^3}{42525} + \frac{210944}{3}L\pi^2T^3\zeta_3^2 \\
&-71680LT^3\zeta_3\zeta_5 + 30720LT^3\zeta_{6,2} + \frac{7872512}{15}L\pi^4T^3\zeta_3 - 1899520L\pi^2T^3\zeta_5 + 867328LT^3\zeta_7 \\
&+\frac{212992}{27}L\pi^6T^3 - \frac{1150976}{15}L\pi^4T^2\zeta_3 + 665600L\pi^2T^2\zeta_5 - 268800LT^2\zeta_7 + \frac{2378752}{405}L\pi^6T^2 \\
&+43008LT^2\zeta_3^2 + \frac{757760}{3}L\pi^2T^2\zeta_3 - 1587200LT^2\zeta_5 - \frac{14838784}{9}L\pi^4T^2 - \frac{2152448L\pi^6T}{2835} \\
&-163840LT\zeta_3^2 + \frac{24051712}{9}L\pi^2T\zeta_3 + 364544LT\zeta_5 + \frac{390412288}{405}L\pi^4T + 2457600LT\zeta_3 \\
&-\frac{39706624}{9}L\pi^2T - \frac{5324800}{9}L\pi^2\zeta_3 + \frac{1998848L\zeta_5}{5} - \frac{34199552L\pi^4}{225} + \frac{9797632L\zeta_3}{3} \\
&+\frac{61534208L\pi^2}{81} - \frac{23560192L}{3} - \frac{11264}{105}\pi^6T^5\zeta_5 + \frac{73216}{5}\pi^4T^5\zeta_7 - 285120\pi^2T^5\zeta_9 \\
&+1271952T^5\zeta_{11} - \frac{10544\pi^{10}T^4}{93555} + \frac{91136}{9}\pi^4T^4\zeta_3^2 - \frac{520832}{3}\pi^2T^4\zeta_3\zeta_5 + 179424T^4\zeta_5^2 \\
&+361088T^4\zeta_3\zeta_7 + \frac{16768}{3}\pi^2T^4\zeta_{6,2} - 26432T^4\zeta_{8,2} + \frac{65536}{45}\pi^6T^4\zeta_3 - 63488\pi^4T^4\zeta_5 \\
&+401408\pi^2T^4\zeta_7 - 508032T^4\zeta_9 + \frac{5137792\pi^6T^3\zeta_3}{2835} - 768T^3\zeta_3^3 + 30976\pi^4T^3\zeta_5 \\
&-\frac{941632}{3}\pi^2T^3\zeta_7 + \frac{2211904T^3\zeta_9}{3} - \frac{142816\pi^8T^3}{14175} + \frac{1183232}{3}\pi^2T^3\zeta_3^2 - 337664T^3\zeta_3\zeta_5 \qquad + \dots
\end{aligned}
$$

Double scaling limit: $\theta \to i\infty,\ g \to 0,$

Ericksson,Semenoff,Zarembo

$$\hat{g} \equiv g \cos\frac{\theta}{2} = \text{fixed}$$

(for some fixed phi)

Selects only ladder diagrams

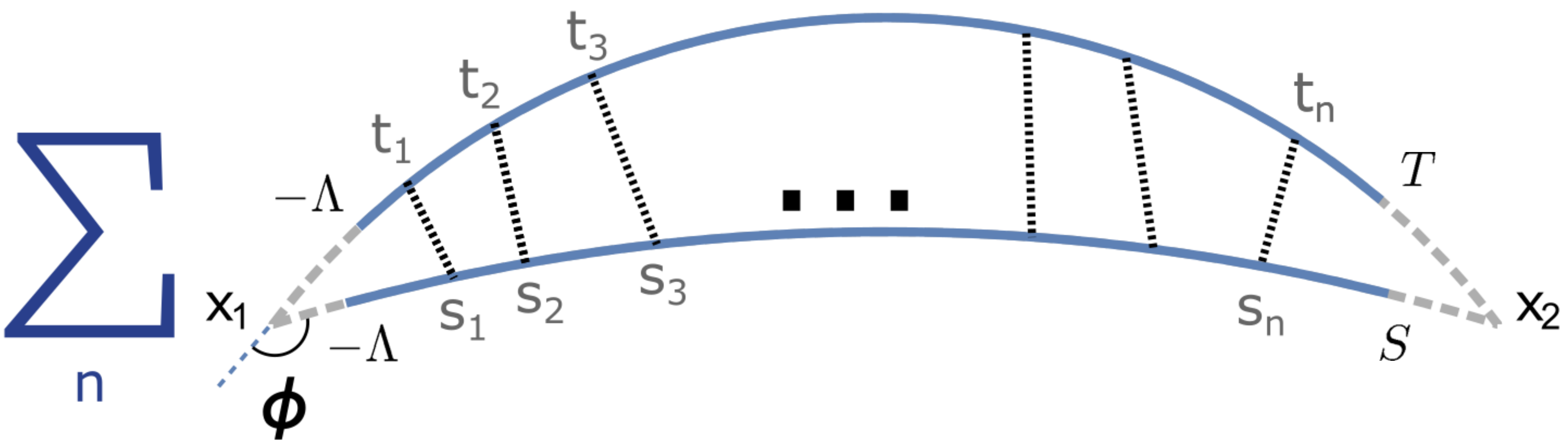
Similar to SYK model

$$T \equiv \frac{1}{\cos^2\frac{\theta}{2}} \to 0$$

$$\begin{aligned}\frac{\Omega}{4\pi} = {} & \hat{g}^2 + \\ & \hat{g}^4\left[16L - 8\right] + \\ & \hat{g}^6\left[128L^2 + L\left(64 + \cancel{\frac{64\pi^2 T}{3}}\right) - 112 - \frac{8\pi^2}{3} + \cancel{72T\zeta_3}\right] + \\ & \hat{g}^8\left[\frac{2048L^3}{3} + \frac{1024}{3}\cancel{\pi^2 L^2 T} + 2048L^2 + \cancel{LT\left(768\zeta_3 + \frac{2176\pi^2}{3}\right)} + \left(-768 - \frac{640\pi^2}{3}\right)L\right. \\ & \left. + \cancel{T^2\left(128\pi^2\zeta_3 - 760\zeta_5\right)} + \cancel{T\left(384\zeta_3 - 640\pi^2 + \frac{32\pi^4}{9}\right)} + \frac{1664\zeta_3}{3} + \frac{1216\pi^2}{9} - 1280\right] + \ldots\end{aligned}$$

$$L \equiv \log\sqrt{8e^{\gamma}\pi\hat{g}^2}$$

$$G(\Lambda, S, T) =$$



$$W = \mathrm{Tr}\,\mathcal{P}\exp\int dt\left[iA\cdot\dot{x} + \vec{\Phi}\cdot\vec{n}\,|\dot{x}|\right]$$

Gauge fields drop out,
only scalars remain

$$\vec{n}\cdot\vec{n}_\theta = \cos\theta \to \infty$$

propagator

$$\partial_S\partial_T\, G = 2\hat{g}^2\frac{|\dot{x}(S)||\dot{x}(T)|}{|x(S)-x(T)|^2}\, G$$

Bethe-Salpeter equation

$$\partial_S \partial_T\, G = 2\hat{g}^2 \frac{|\dot{x}(S)||\dot{x}(T)|}{|x(S)-x(T)|^2}\, G$$

$$\Downarrow$$

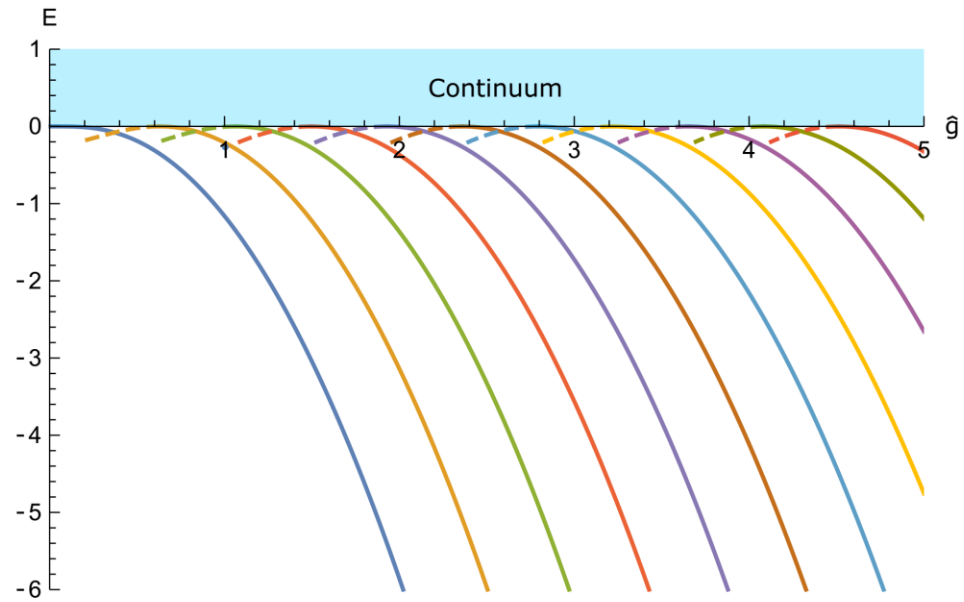
$$\frac{1}{4}\partial_y^2 \tilde{G} = \partial_x^2 \tilde{G} + \frac{2\hat{g}^2}{\cosh x + \cos\phi}\tilde{G}$$

$$x = S - T,\ \ y = \frac{S+T}{2}$$

$$G = \sum_n c_n F_n(x) e^{-\sqrt{-E_n}\, y}$$

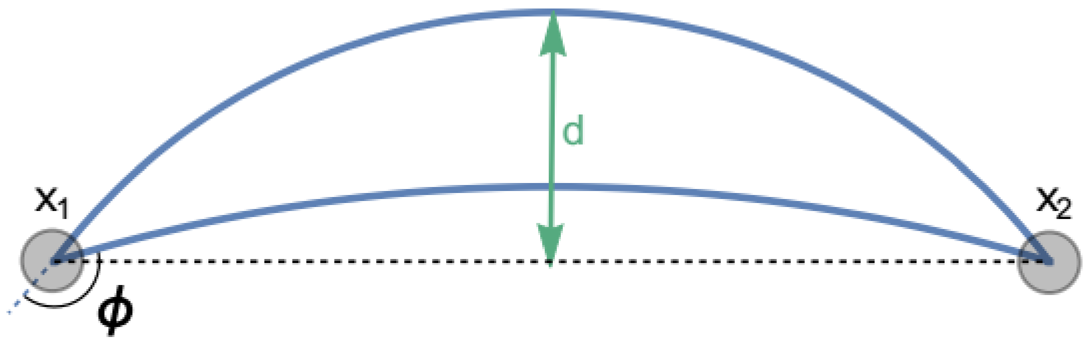
We get Schrodinger equation

$$\boxed{\left[\partial_x^2 - \frac{2\hat{g}^2}{\cosh x + \cos\phi}\right] F_n(x) = \frac{1}{4} E_n F_n(x)}$$



Scaling dimension $\longleftrightarrow$ Ground state energy

$$\Delta_0 = -\sqrt{-E}$$



So the 2-pt function is $\dfrac{2F_0^2(0)}{\Delta_0}\left(\dfrac{\epsilon}{x_{12}}\right)^{2\Delta_0}$

We see the correct conformal spacetime dependence

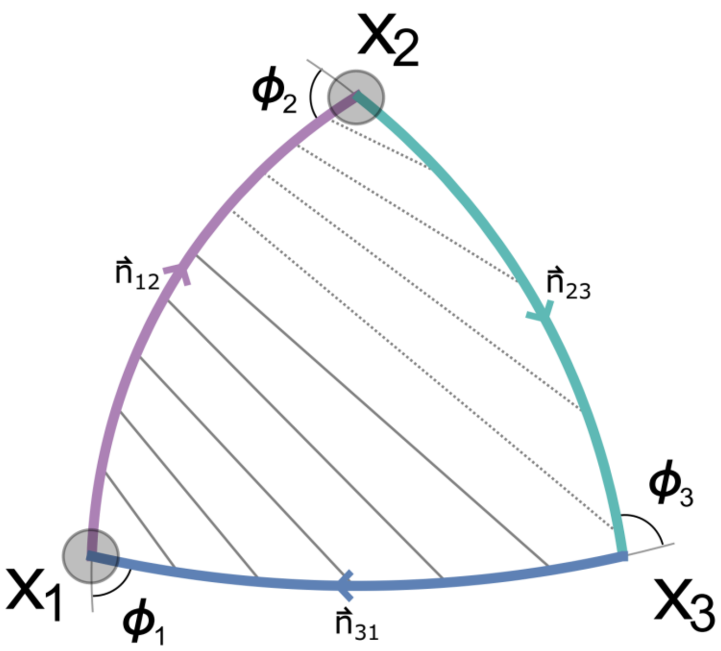
Ericksson, Semenoff, Zarembo 2000
Cavaglia, Gromov, FLM 2018

Next: 3-point correlator

$$W_{123}^{\bullet\bullet\circ,\epsilon} = \frac{\epsilon^{\Delta_1+\Delta_2}}{x_{12}^{\Delta_1+\Delta_2}\, x_{13}^{\Delta_1-\Delta_2}\, x_{23}^{\Delta_2-\Delta_1}} (L_{123})^{\Delta_1} (L_{231})^{\Delta_2} \left(c_0|_{\Delta_1,\phi_1} \right) \left(c_0|_{\Delta_2,\phi_2} \right) \mathcal{N}_{123}^{\bullet\bullet\circ},$$

Where

$$\mathcal{N}_{123}^{\bullet\bullet\circ} = 2\hat{g}_1^2 \int_{-\infty}^{0} ds \int_{-\infty}^{0} dt\, \frac{F_{\Delta_1,\phi_1}(-\delta x_1 + s - t)\, F_{\Delta_2,\phi_2}(-\delta x_2 - T_{12}(s))\, e^{-\frac{s+t}{2}\Delta_1 - \frac{T_{12}(s)}{2}\Delta_2}}{\cosh(s-t-\delta x_1) + \cos\phi_1}$$



Where

$$e^{T_{12}(s)} = \frac{(1-e^s)}{1 - e^s \frac{\cos\phi_3 - \cos(\phi_1+\phi_2)}{\cos\phi_3 - \cos(\phi_1-\phi_2)}}.$$

Where

$$L_{123} = \frac{\sqrt{\sin\frac{1}{2}(\phi_1+\phi_2-\phi_3)\, \sin\frac{1}{2}(\phi_1-\phi_2+\phi_3)}}{\sin\phi_1}$$

Where

$$\delta x_1 = \log \frac{\sin\left(\frac{1}{2}(\phi_1-\phi_2+\phi_3)\right)}{\sin\left(\frac{1}{2}(\phi_1+\phi_2-\phi_3)\right)}$$

Where

$$c_n = -\frac{2F_n(0)}{\Delta_n}$$

We keep two couplings $\hat{g}_1, \ \hat{g}_2$

while $\hat{g}_3 = 0$

$$\hat{g}_k = g\cos\frac{\theta_k}{2}$$

How to simplify this mess ???

LET’S USE INTEGRABILITY

Ladder limit: only scalar fields survive, half of QSC disappears

$$\mathbf{Q}_i^{[+4]} D_0 - \mathbf{Q}_i^{[+2]} \left[D_1 - \mathbf{P}_a^{[+2]} \mathbf{P}^{a[+4]} D_0 \right] + \mathbf{Q}_i \left[D_2 - \mathbf{P}_a \mathbf{P}^{a[+2]} D_1 + \mathbf{P}_a \mathbf{P}^{a[+4]} D_0 \right]$$
$$- \mathbf{Q}_i^{[-2]} \left[\bar{D}_1 + \mathbf{P}_a^{[-2]} \mathbf{P}^{a[-4]} \bar{D}_0 \right] + \mathbf{Q}_i^{[-4]} \bar{D}_0 = 0$$

P-functions become trivial and the general Baxter equation reduces to

$$\boxed{\left(-2u^2 \cos\phi + 2\Delta u \sin\phi + 4\hat{g}^2\right) q(u) + u^2 q(u-i) + u^2 q(u+i) = 0}$$

$$q_i(u) = \mathbf{Q}_i(u)/\sqrt{u}$$

$$q_1 \sim M_1 e^{\phi u} u^{\Delta}, \quad q_4 \sim M_4 e^{-\phi u} u^{-\Delta}, \quad u \to \infty$$

Quantization condition (comes from analyticity of QSC):

$$\boxed{\Delta = -\frac{2\hat{g}^2}{\sin\phi} \frac{q_1(0)\bar{q}_1'(0) + \bar{q}_1(0) q_1'(0)}{q_1(0)\bar{q}_1(0)}}$$

Cavaglia, Gromov, FLM 2018

Direct relation with the wave functions by Mellin transform

$$F(z) = -i\, e^{-\Delta z/2} \int_{c-i\infty}^{c+i\infty} \frac{q_1(u)}{u}\, e^{w_\phi(z)\, u}\, du, \quad c > 0, \quad \text{where} \quad e^{i w_\phi(z)} = \left(\frac{\cosh \frac{z - i\phi}{2}}{\cosh \frac{z+i\phi}{2}} \right)$$

Baxter equivalent to Schrodinger

Now we plug this into the result we got from diagrams

$$\mathcal{N}_{123}^{\bullet\bullet\circ} = 2\hat{g}_1^2 \int_{-\infty}^{0} ds \int_{-\infty}^{0} dt\, \frac{F_{\Delta_1,\phi_1}(-\delta x_1 + s - t)\, F_{\Delta_2,\phi_2}(-\delta x_2 - T_{12}(s))\, e^{-\frac{s+t}{2}\Delta_1 - \frac{T_{12}(s)}{2}\Delta_2}}{\cosh(s - t - \delta x_1) + \cos\phi_1}$$

The structure constant simplifies drastically!

This is our main result

q-function

for zero coupling

$$\boxed{C_{123}^{\bullet\bullet\circ} = (K_{123})^{\Delta_1} (K_{213})^{\Delta_2} \frac{\int_{|} q_1\, q_2\, e^{-\phi_3 u} \frac{du}{2\pi i u}}{\sqrt{\int_{|} q_1\, q_1 \frac{du}{2\pi i u}} \sqrt{\int_{|} q_2\, q_2 \frac{du}{2\pi i u}}},} \qquad K_{123} = \frac{\sin \frac{1}{2}(\phi_1 + \phi_2 - \phi_3)}{\sin \phi_1}$$

Get scalar product of 3 q-functions – precisely the kind of result expected from separation of variables

At 1 loop we reproduce the result of a direct calculation

$$(C^{\bullet\bullet\circ}) \qquad = 1 + \hat{g}_1^2 F_{123} + \hat{g}_2^2 F_{213} + \ldots$$

$$F_{123} = \frac{1}{\sin\phi_1}\Big[2i\left(\mathrm{Li}_2(e^{-2i\phi_1}) - \mathrm{Li}_2(e^{-i\phi_1-i\phi_2+i\phi_3}) + \mathrm{Li}_2(e^{i\phi_1-i\phi_2+i\phi_3})\right) - \frac{i\pi^2}{3}$$
$$+2\left(\phi_1-\phi_2+\phi_3\right)\log\left(\frac{1-e^{-i\phi_1-i\phi_2+i\phi_3}}{1-e^{i\phi_1-i\phi_2+i\phi_3}}\right) - 4\phi_1\log\left(\frac{\sin\frac{1}{2}\left(\phi_1+\phi_2-\phi_3\right)}{\sin\phi_1}\right)\Big]$$

In the limit of straight lines we get

$$\langle q_1 q_2 e^{-\phi_3 u} \rangle = \frac{1}{\Gamma(-\Delta_1 - \Delta_2 + 1)}$$

$$\langle f(u) \rangle \equiv \left(2 \sin \frac{\beta}{2}\right)^{\alpha} \int_{c-i\infty}^{c+i\infty} f(u) \frac{du}{2\pi i u} \quad , \quad c > 0$$

$$C_{123}^{\bullet\bullet\circ}\big|_{\phi_i=0} = \frac{\sqrt{\Gamma(1-2\Delta_1)\,\Gamma(1-2\Delta_2)}}{\Gamma(1-\Delta_1-\Delta_2)}$$

Matches direct all-loop calculation of [Kim, Kiryu, Komatsu, Nishimura 2017] !

Would be interesting to compare with other approaches for 3pt functions (e.g. “hexagons”)

[Basso, Komatsu, Vieira 15] [Fleury, Komatsu 16]

Very recently: expect integrability beyond large Nc as well !

[Caetano, Bargheer, Fleury, Komatsu, Vieira 17]

Rewriting the Baxter equation in operator form:

$$\hat{O} \equiv \frac{1}{u}\left[(4\hat{g}^2 - 2u^2\cos\phi + 2\Delta u \sin\phi) + u(u-i)D^{-1} + u(u+i)D\right]\frac{1}{u} \qquad \hat{O}q(u) = 0\,.$$

We find that the operator is “self-adjoint”:

$$\int_| q_1(u)\hat{O}q_2(u)du = \int_| q_2(u)\hat{O}q_1(u)du$$

Immediate consequences:

- Two solutions with different Δ are orthogonal: $0 = \int_| q_1(u)(\hat{O}_1 - \hat{O}_2)q_2(u)du = (\Delta_1 - \Delta_2)2\sin\phi \int_| \frac{q_1(u)q_2(u)}{u}du$
- Closed equation for the derivative of Δ w.r.t. the coupling: $-\frac{1}{4}\frac{\partial\Delta}{\partial\hat{g}^2} = \frac{\langle q^2\frac{1}{u}\rangle}{\langle q^2\rangle}$

Can be considered as a correlator of two cusps with Lagrangian, has very similar form to the 3-cusp correlator!

$$\langle f(u)\rangle \equiv \left(2\sin\frac{\beta}{2}\right)^{\alpha}\int_{c-i\infty}^{c+i\infty} f(u)\frac{du}{2\pi i u}\ , \quad c>0$$

EXCITED STATES

excited states $\longleftrightarrow$ local operators (scalars) inserted at the cusps

Cavaglia, Gromov, FLM 2018

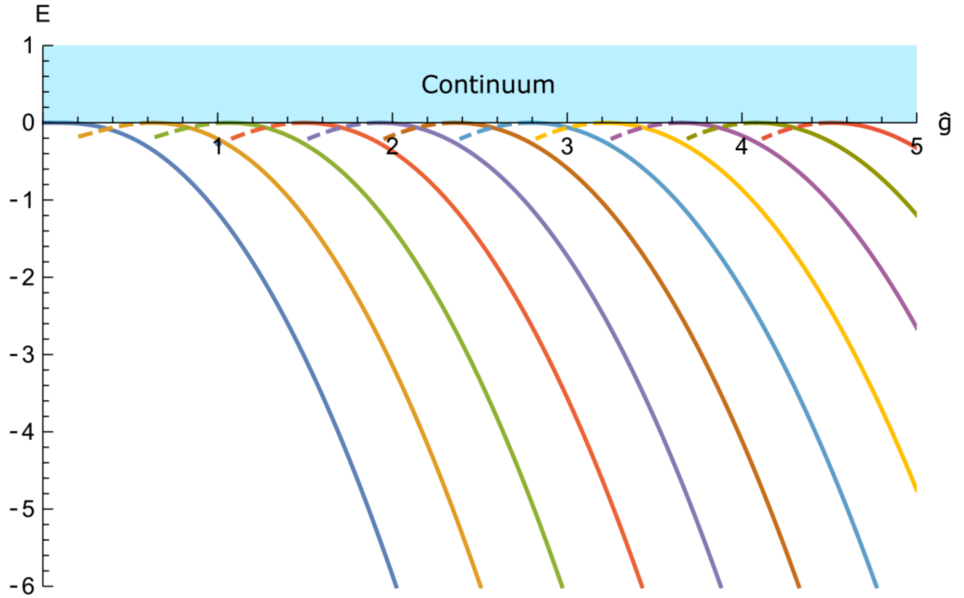
At weak coupling we found:

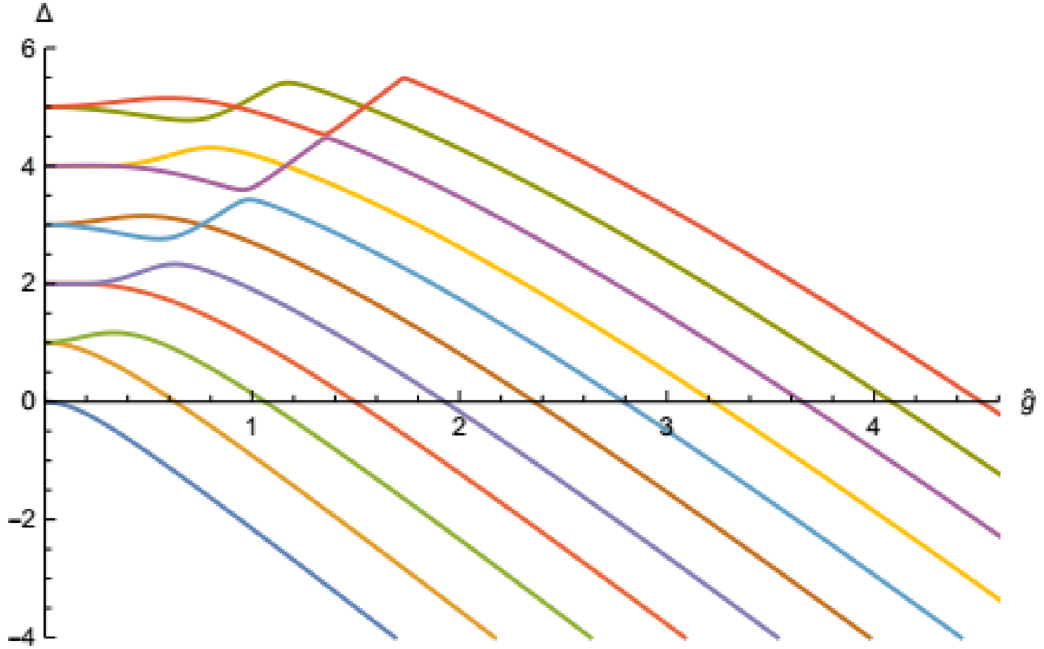
$$\Delta_{L,\pm} = L \pm \frac{4}{L}\frac{\sin L\phi}{\sin\phi}\hat{g}^2 + \dots ,$$

At strong coupling we found:

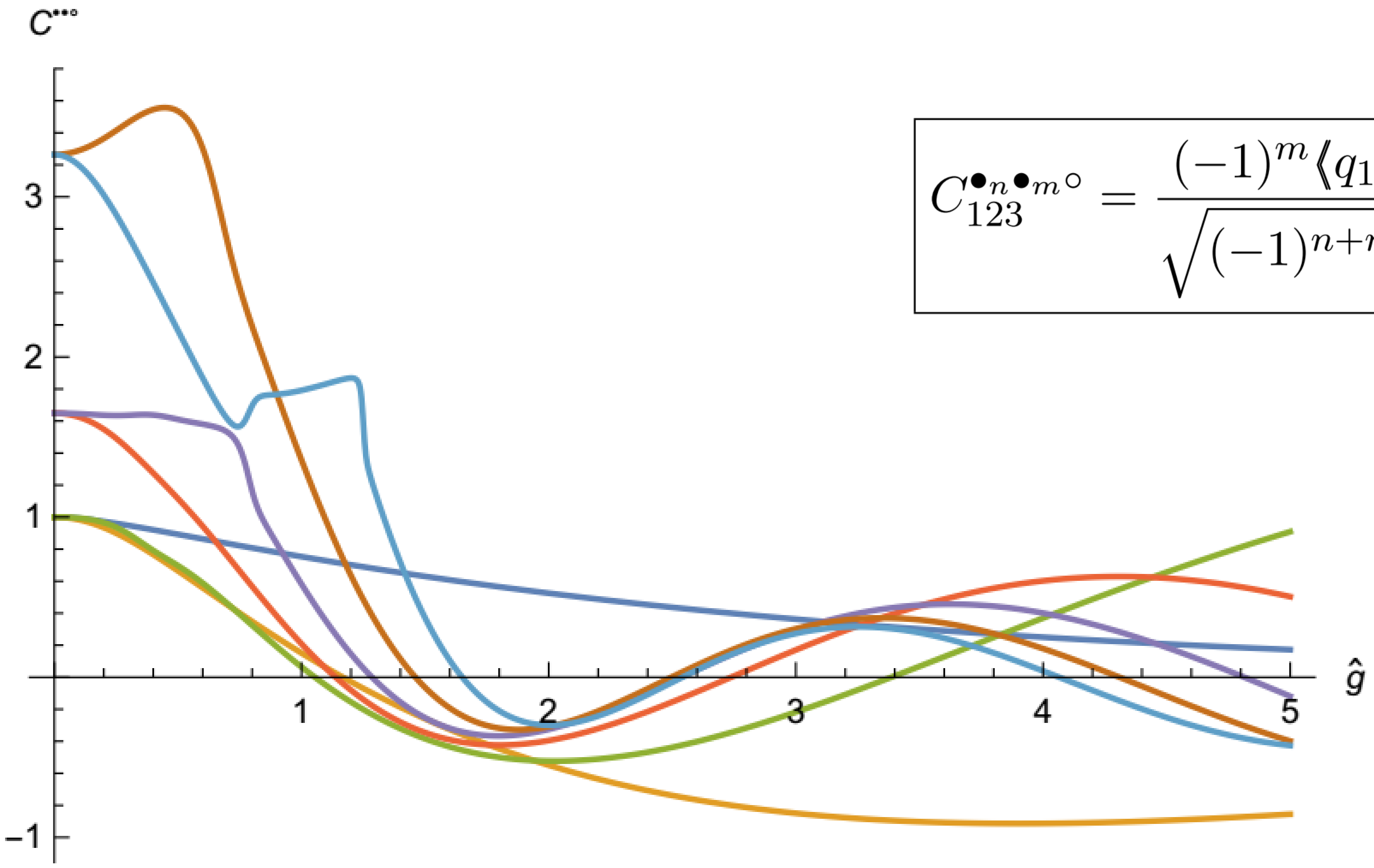
$$\frac{\Delta_n}{2\sec\left(\frac{\phi}{2}\right)} = -g + \frac{2n+1}{4} + \frac{(2n^2+2n+1)\cos\phi - 2n^2 - 2n - 3}{64g}$$
$$+ (2n+1)\sin^2\left(\frac{\phi}{2}\right)\frac{(n^2+n+1)\cos\phi - n^2 - n + 11}{512g^2} + \mathcal{O}\left(g^{-3}\right)$$

Matches perturbation theory
[Brüser, Caron-Huot, Henn 2018] at two loops



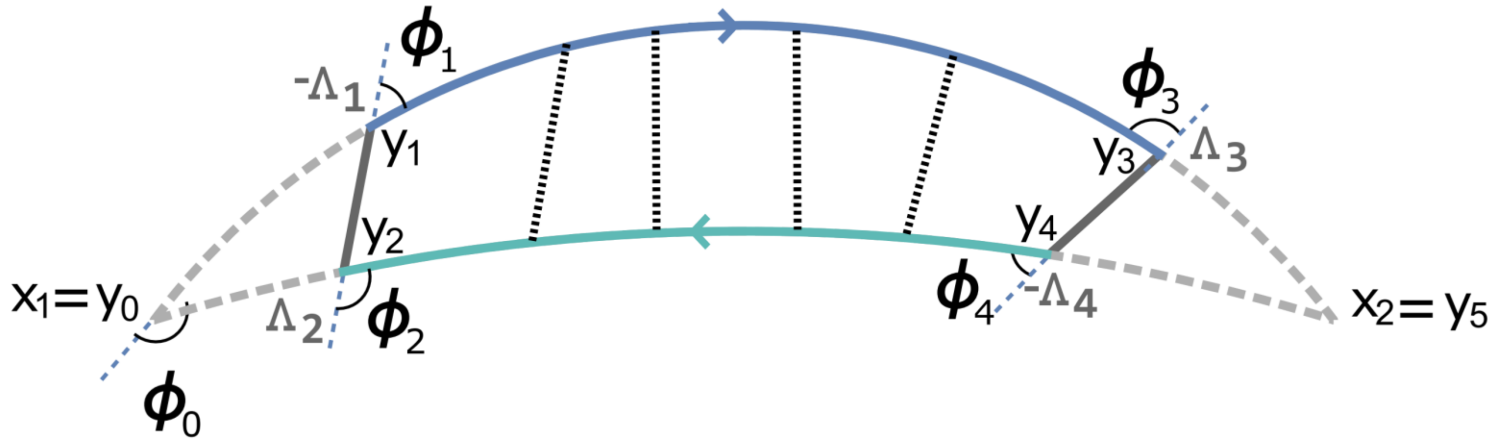


The 3-cusp correlator is given by just the same formula!



$$C_{123}^{\bullet_n \bullet_m \circ} = \frac{(-1)^m \langle\!\langle q_{1,n} q_{2,m} e^{-\phi_3 u} \rangle\!\rangle}{\sqrt{(-1)^{n+m} \langle\!\langle q_{1,n}^2 \rangle\!\rangle \langle\!\langle q_{2,m}^2 \rangle\!\rangle}}$$

Cavaglia, Gromov, FLM 2018



$$G(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = \sum_{n=0}^{\infty} C_{012}^{\bullet n \circ\circ} C_{043}^{\bullet n \circ\circ} \left(\frac{e^{-2\Lambda}}{L_{043} L_{012}} \right)^{\Delta_n}$$

We can identify structure constants in the 4pt function

Crossing equation? Conformal bootstrap?

$$L_{abc} = \frac{\sqrt{\sin\frac{1}{2}(\phi_a + \phi_b - \phi_c)\,\sin\frac{1}{2}(\phi_a - \phi_b + \phi_c)}}{\sin\phi_a}$$

INTEGRABLE FISHNET THEORY

We worked in the ladders limit for the Wilson line

A very similar limit can be realized at the level of local operators

Gamma-deformed N=4 SYM:

$$\mathcal{L}_{\text{int}} = N_c\, g^2 \,\text{tr} \left(\frac{1}{4}\{\phi_i^\dagger, \phi^i\}\{\phi_j^\dagger, \phi^j\} - e^{-i\epsilon^{ijk}\gamma_k}\phi_i^\dagger\phi_j^\dagger\phi^i\phi^j \right) + \text{gauge fields} + \text{fermions}$$

$\gamma_1,\ \gamma_2,\ \gamma_3$ are 3 deformation parameters

Remains conformal and integrable

Beisert, Roiban 05
Gromov, FLM 10
Arutyunov, de Leeuw, Tongeren 13
Kazakov, Leurent, Volin 15

Double scaling limit: strong twist, weak coupling

$$g \to 0, \qquad e^{-i\gamma_j/2} \to \infty, \qquad \xi_j = g\, e^{-i\gamma_j/2} - \text{fixed}, \qquad (j = 1, 2, 3.)$$

Just like the ladders limit

Result known as fishnet theory

Gauge fields decouple, remain only scalars + fermions

Gurdogan, Kazakov 15

$$\mathcal{L}_{\text{int}} = N_c\, \text{tr}[\xi_1^2\, \phi_2^\dagger\phi_3^\dagger\phi_2\phi_3 + \xi_2^2\, \phi_3^\dagger\phi_1^\dagger\phi_3\phi_1 + \xi_3^2\, \phi_1^\dagger\phi_2^\dagger\phi_1\phi_2 +$$
$$+ i\sqrt{\xi_2\xi_3}(\psi^3\phi^1\psi^2 + \bar{\psi}_3\phi_1^\dagger\bar{\psi}_2) + i\sqrt{\xi_1\xi_3}(\psi^1\phi^2\psi^3 + \bar{\psi}_1\phi_2^\dagger\bar{\psi}_3) + i\sqrt{\xi_1\xi_2}(\psi^2\phi^3\psi^1 + \bar{\psi}_2\phi_3^\dagger\bar{\psi}_1)\,].$$

No susy but still solvable !

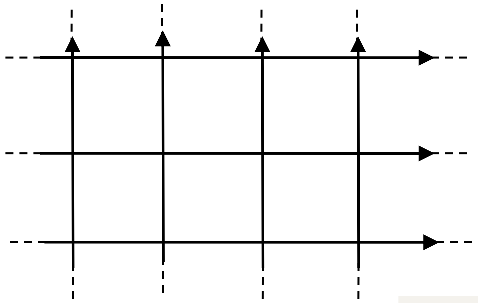
(at large N)

Simplest case: theory with two N x N scalar fields

$$\mathcal{L}[\phi_1,\phi_2] = \frac{N}{2}\mathrm{tr}\ \left(\partial^\mu\phi_1^\dagger\partial_\mu\phi_1 + \partial^\mu\phi_2^\dagger\partial_\mu\phi_2 + 2\xi^2\ \phi_1^\dagger\phi_2^\dagger\phi_1\phi_2\right)\ .$$

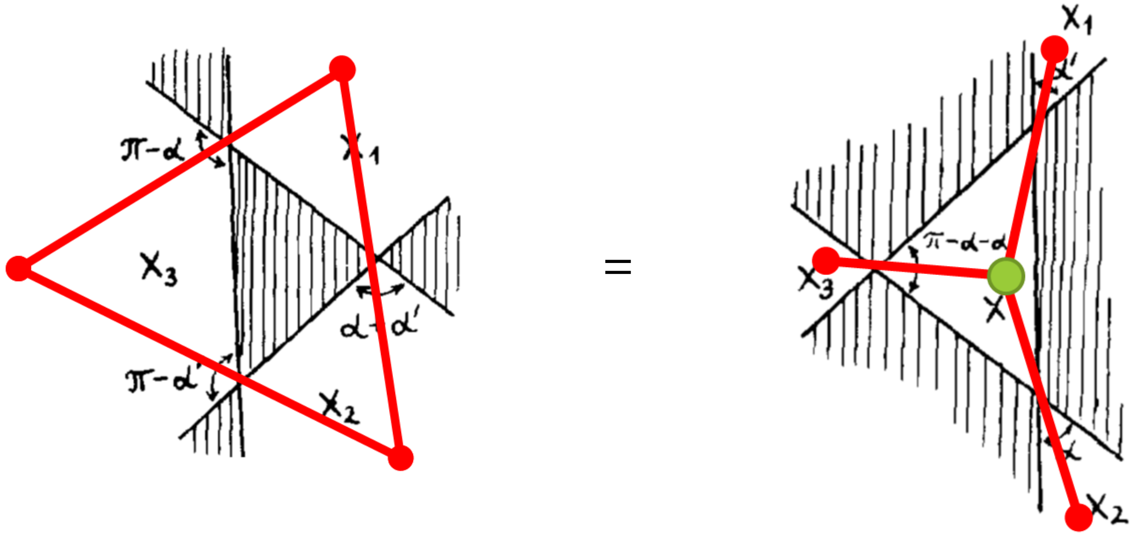
Baby version of N=4 SYM   Gurdogan, Kazakov 15

Observables given by “fishnet” graphs



Zamolodchikov 81

Yang-Baxter equation at the level of diagrams



Possible due to star-triangle relation with m=0 propagators

$$\int\frac{d^Dx_0}{|x_{10}|^{2a}|x_{20}|^{2b}|x_{30}|^{2c}} = \frac{V(a,b,c)}{|x_{12}|^{D-2c}|x_{23}|^{D-2a}|x_{31}|^{D-2b}}\,,$$

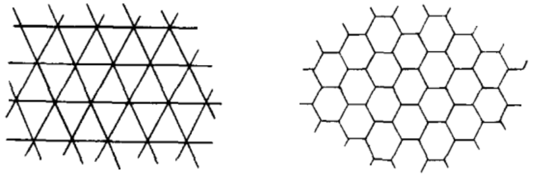
$$a+b+c=D\,,\ \ x_{ij}\equiv x_i-x_j$$

$$V(a,b,c)=\pi^{D/2}\frac{\Gamma(\frac{D}{2}-a)\Gamma(\frac{D}{2}-b)\Gamma(\frac{D}{2}-c)}{\Gamma(a)\Gamma(a)\Gamma(c)}$$

Can use integrability to compute individual Feynman graphs

Unique possibility to see origins of integrability in gauge theory

Analogs are known in 3d and 6d

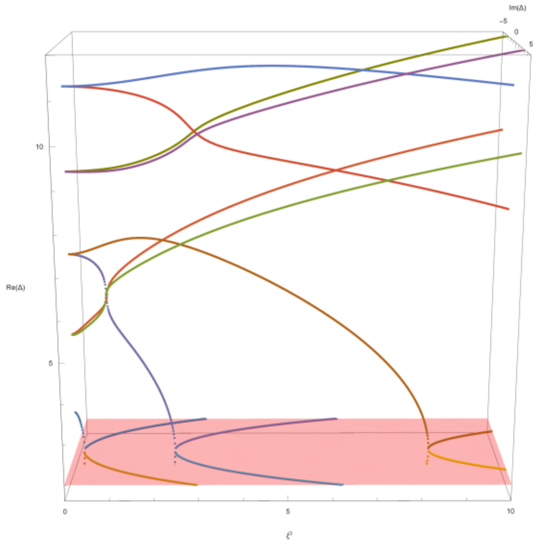


QSC is inherited from N=4 SYM

Previously developed methods [Gromov, FLM, Sizov 15] are very efficient for spectrum

$$
\begin{aligned}
\Delta_3 - 3 = & -12\zeta_3\xi^6 + \left(189\zeta_7 - 144\zeta_3{}^2\right)\xi^{12} \\
& +\xi^{18}\left(-1944\zeta_{8,2,1} - 3024\zeta_3{}^3 - 3024\zeta_5\zeta_3{}^2 + 6804\zeta_7\zeta_3 + \frac{198\pi^8\zeta_3}{175} + \frac{612\pi^6\zeta_5}{35} + 270\pi^4\zeta_7 + 5994\pi^2\zeta_9 - \frac{925911\zeta_{11}}{8}\right) \\
& +\xi^{24}\left(-93312\zeta_3\zeta_{8,2,1} + \frac{10368}{5}\pi^4\zeta_{8,2,1} + 5184\pi^2\zeta_{9,3,1} + 51840\pi^2\zeta_{10,2,1} - 148716\zeta_{11,3,1} - 1061910\zeta_{12,2,1}\right. \\
& +62208\zeta_{10,2,1,1,1} - 77760\zeta_3{}^4 - 145152\zeta_5\zeta_3{}^3 - \frac{576}{7}\pi^6\zeta_3{}^3 - 864\pi^4\zeta_5\zeta_3{}^2 - 2592\pi^2\zeta_7\zeta_3{}^2 + 244944\zeta_7\zeta_3{}^2 \\
& +186588\zeta_9\zeta_3{}^2 + \frac{9504}{175}\pi^8\zeta_3{}^2 - 2592\pi^2\zeta_5{}^2\zeta_3 + \frac{29376}{35}\pi^6\zeta_5\zeta_3 + 298404\zeta_5\zeta_7\zeta_3 + 12960\pi^4\zeta_7\zeta_3 + 287712\pi^2\zeta_9\zeta_3 \\
& -5555466\zeta_{11}\zeta_3 + \frac{2910394\pi^{12}\zeta_3}{2627625} + 57672\zeta_5{}^3 - 71442\zeta_7{}^2 + \frac{13953\pi^{10}\zeta_5}{1925} + \frac{7293\pi^8\zeta_7}{175} - \frac{19959\pi^6\zeta_9}{5} \\
& \left.+\frac{119979\pi^4\zeta_{11}}{2} + \frac{10738413\pi^2\zeta_{13}}{2} - \frac{4607294013\zeta_{15}}{80}\right) + O\left(\xi^{30}\right) \qquad (1.7)
\end{aligned}
$$

Gromov, Kazakov, Korchemsky, Negro, Sizov 17



For operator $\mathrm{tr}(\phi_1\phi_1)$ get all-loop result

Grabner, Gromov, Kazakov, Korchemsky 17

$$(\Delta-4)(\Delta-2)^2\Delta = 16\xi^4 \qquad C_{123} = \frac{1}{\pi^4(-\Delta^2+4\Delta-2)}\frac{\Gamma(-\Delta+4)\Gamma^2(\Delta/2)}{\Gamma^2(-\Delta/2+4)\Gamma(\Delta-1)}$$

Extension to theory with scalars + fermions is underway

Kazakov, FLM, Olivucci, Preti in progress

Very similar to Wilson lines correlators! $C_{123}^{\bullet\bullet\circ}|_{\phi_i=0} = \frac{\sqrt{\Gamma(1-2\Delta_1)\,\Gamma(1-2\Delta_2)}}{\Gamma(1-\Delta_1-\Delta_2)}$

Hope to see similar structure,
i.e. scalar product of Q-functions [in progress]

Dual string model - ???

CONCLUSIONS

- All-loop C_{123} strikingly simplifies in terms of Q-functions
- A lot to do: beyond ladders, more insertions, fishnet theory, …
- Related structures seen in localization [Giombi, Komatsu 18]
- Surprising simplifications for SU(N) spin chains give extra hope [Gromov, FLM, Sizov 16,18]
- Algebraic / rep theory interpretation of $\int q_1 q_2 q_3$?
- Can we bootstrap the full 3-pt structure constant ?