New large N solutions in the Sachdev-Ye-Kitaev model

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based on work in progress with I. Ya. Aref'eva, M. A. Khramtsov and I. V. Volovich

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The Sachdev-Ye-Kitaev (SYK) model

The SYK Hamiltonian

$$H_{SYK} = j_{abcd}\psi_a\psi_b\psi_c\psi_d$$

where j_{abcd} are constants taken from the Gaussian distribution

$$\langle j_{abcd}^2 \rangle = \frac{J^2}{N^3}$$

and ψ are the Majorana fermions satisfying

$$\{\psi_a, \psi_b\} = \delta_{ab}, \quad a, b = 1, ..., N$$

Free energy and replica trick

In order to evaluate the free energy in a quenched disorder system, one needs to use the replica trick

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Take M replicas of the system

$$Z(\beta)^M = \int D\psi e^{-\sum_{\alpha=1}^{M} \int_0^\beta i\dot{\psi}^\alpha(\tau)\psi^\alpha(\tau) + j_{abcd}\psi_a^\alpha(\tau)\psi_b^\alpha(\tau)\psi_c^\alpha(\tau)\psi_d^\alpha(\tau)}$$

To average over disorder, we introduce bilocal fields G and Σ in order to simplify the calculations. Σ is a Lagrange multiplier that sets $G_{\alpha\beta}(\tau, \tau') = \frac{1}{N} \sum_a \psi_a^\alpha(\tau)\psi_a^\beta(\tau')$.

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$$\overline{Z(\beta)^M} = \int DGD\Sigma \; \mathrm{Det}[\,\delta_{\alpha\beta}\partial_\tau - \Sigma_{\alpha\beta}]^{\frac{N}{2}} \times$$

$$\exp\left[-\frac{N}{2}\sum_{\alpha\beta}\int_0^\beta\int_0^\beta d\tau_1 d\tau_2 \left(\Sigma_{\alpha\beta}(\tau_1,\tau_2)G_{\alpha\beta}(\tau_1,\tau_2) - \frac{J^2}{4}G_{\alpha\beta}(\tau_1,\tau_2)^4\right)\right]$$

saddle point eqs: $G(\omega) = \frac{1}{-i\omega - \Sigma(\omega)}$, $\Sigma_{\alpha\beta}(\tau,\tau') = J^2 G_{\alpha\beta}(\tau,\tau')^3$

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and the saddle point equations:

$$G(\omega) = -\frac{1}{\Sigma(\omega)}, \quad \Sigma_{\alpha\beta}(\tau,\tau') = J^2 G_{\alpha\beta}(\tau,\tau')^3$$

Replica Diagonal (RD) solution and Replica Symmetry Breaking (RSB)

In the IR limit the saddle point equations are exactly solvable

$$G_{\alpha\beta}(\tau) = P_{\alpha\beta} b \frac{\operatorname{sgn}(\tau)}{|J\tau|^{2\Delta}}, \quad \Delta = \frac{1}{4}$$

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  - $P_{\alpha\beta}$ = Parisi martix [present work]

Parisi martices

Parisi matrices are constructed of m_i blocks, where $i = 1, ..., M$ and l parameters under the following conditions [Parisi'72]:

$$P_{l+1} = \begin{pmatrix} P_l & a_l \mathcal{J}_l & ... & a_l \mathcal{J}_l \\ a_l \mathcal{J}_l & P_l & ... & a_l \ \mathcal{J}_l \\ & ... & ... & \\ a_l \mathcal{J}_l & a_l \mathcal{J}_l & ... & P_l \end{pmatrix}, \quad \mathcal{J}_l = \begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & 1 & ... & 1 \\ & .. & ... & \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

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In the case of $M = 4$ it looks like

$$P_4 = \begin{pmatrix} a_0 & a_1 & a_2 & a_2 \\ a_1 & a_0 & a_2 & a_2 \\ a_2 & a_2 & a_0 & a_1 \\ a_2 & a_2 & a_1 & a_0 \end{pmatrix}$$

Parisi martices

The saddle point equations for the parameters of the matrix

$$\mathcal{C} = a_0^4 + \sum a_j^4 (m_{j+1} - m_j)$$

$$a_j a_0^3 + a_0 a_j^3 + \sum (a_i a_j^3 + a_j a_i^3)(m_{i+1} - m_i) = m_j a_j^4 - \sum a_i^4 (m_{i+1} - m_i)$$

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In the limit $M \to 0$ the saddle point eqs become

$$\mathcal{C} = a_0^4 - \langle a^4 \rangle$$

$$a(u)[a_0^3 - \langle a^3 \rangle] + a^3(u)[a_0 - \langle a \rangle] = \int_0^u [a(v) - a(u)][a^3(v) - a^3(u)]\, dv$$

where the angular brackets denote the average over v

$$\int_0^1 a^p(v)\, dv \equiv \langle a^p \rangle$$

One-step ansatz

To solve the equations, we need to make some assumption. Let the function $a(u)$ be:

$$a(u) = A_0 + A_1\theta(u - \mu)$$

where μ is the breakpoint and which is considered in [Georges,Parcollet,Sachdev'99] in the context of the original SY-model

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The saddle point equations

$$a_0^4 - (A_0 + A_1)^4 + A_1 (2A_0 + A_1) \left(2A_0^2 + 2A_1 A_0 + A_1^2\right) \mu = \mathcal{C}$$

$$\mathrm{A}_0(a_0 A_0^2 + a_0^3 + A_1^3(\mu - 1) + 3A_0 A_1^2(\mu - 1) + 4A_0^2 A_1(\mu - 1) - 2A_0^3) = 0;$$

$$A_1(A_1^2 (a_0 + A_0(3\mu - 7)) + 3A_0 A_1 (a_0 + A_0(\mu - 3)) + 3a_0 A_0^2$$

$$+a_0^3 + A_1^3(\mu - 2) - 4A_0^3) = 0$$

Classes of solutions for $a(u)$

The saddle point equations admit solutions

1. Replica-diagonal solution: $A_0 = 0$, $A_1 = 0$.
2. Replica-symmetric solution: $A_0 = a_0$, $A_1 = 0$.
   It cannot be a solution of the full SYK model, since it requires $\mathcal{C} = 0$.
3. Replica-symmetric complex-valued solutions:
   $$A_0 = \frac{1}{4}\left(-1 \pm i\sqrt{7}\right)$$
   $$A_1 = 0$$
4. RSB simple solution: $A_0 = 0$ which provides
   $$A_1^2 a_0 + a_0^3 + A_1^3(\mu - 2) = 0$$
   $$a_0^3 + a_0 A_1^2 + A_1^3(\mu - 2) = 0$$
5. More complex solutions: $A_0 \neq 0$ and $A_1 \neq 0 \Rightarrow$ numeric calculations

Free energy on the RSB solution

The free energy is:

$$-\beta F = \lim_{M\to 0} \frac{1}{M} \log \overline{Z^M}$$

The replica and time dependencies separate

$$\frac{2}{N} S_M = -\mathrm{Tr}\log[-\Big(\Sigma_{\alpha\beta}\Big)]$$

$+\int_0^\beta \int_0^\beta d\tau_1 d\tau_2 \sum_{\alpha\beta} \Big(\Sigma_{\alpha\beta}(\tau_1,\tau_2) G_{\alpha\beta}(\tau_1,\tau_2) - \frac{J^2}{q}(G_{\alpha\beta}(\tau_1,\tau_2))^q\Big) =$
-M $\log\det[-J^2 g_c(\tau,\tau')^{q-1}] - \Lambda \Big(\log\det[\mathcal{C}^{1/q-1} Q^{\circ(q-1)}] - \frac{1-q}{q} M\Big)$
where $\Lambda \sim \beta J/\pi$. The only one term we have to think about in the $M \to 0$ limit gives:

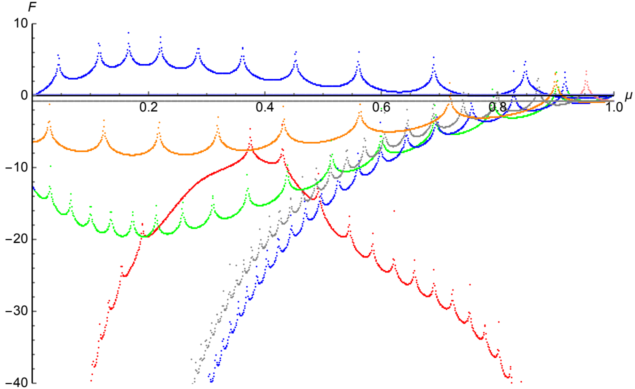
$$\log\det[\mathcal{C}^{1/q-1} Q^{\circ(q-1)}] = \frac{1-q}{q} \log \mathcal{C} + \log(a_0^{q-1} - \langle a^{q-1}\rangle)$$

- $\int_0^1 \frac{dv}{v^2} \log \frac{a_0^{q-1} - \langle a^{q-1}\rangle - [a^{q-1}](v)}{a_0^{q-1} - \langle a^{q-1}\rangle} + \frac{a^{q-1}(0)}{a_0^{q-1} - \langle a^{q-1}\rangle}$

other terms are the same as in the RD case

Free energy on the RSB solution

The plot of the replica dependent term of the free energy as a function of breakpoint at $a_0 = 0.7$ and $\beta J \sim 10$



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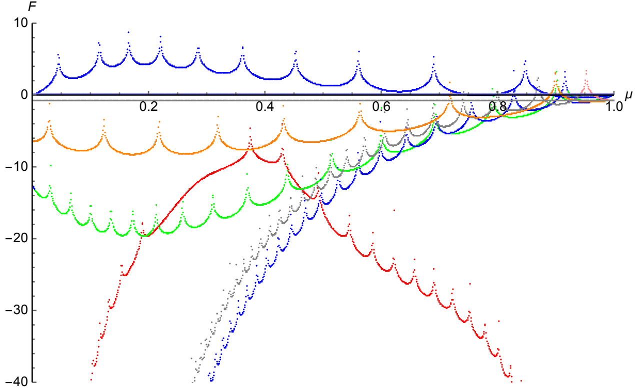

1. saddle points are complex valued
2. a landscape of metastable states

Comparison of RSB solution with replica-diagonal solution

The expression for the free energy:

- RD case low temperature expansion [Maldacena,Stanford'16;Kitaev'17]:

$$\beta F = \beta E_0 - s_0 - \frac{c}{2\beta}$$

E_0 is the vacuum energy, s_0 - zero temperature entropy, $c \sim N/J$ is the specific heat

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- RSB case

$$\beta F = \beta E_0 + \beta E_0^{RSB} - s_0 - \frac{c}{2\beta}$$

  where E_0^{RSB} arises from replicas.

Reparametrization invariance

Reparametrization invariance of the system changes from the replica-diagonal case

$$G_{\alpha\beta}(\tau,\tau') = f'_\alpha(\tau)^\Delta f'_\beta(\tau')^\Delta G_{\alpha\beta}(f_\alpha(\tau), f_\beta(\tau'))$$
$$\Sigma_{\alpha\beta}(\tau,\tau') = f'_\alpha(\tau)^{1-\Delta} f'_\beta(\tau')^{1-\Delta} \Sigma_{\alpha\beta}(f_\alpha(\tau), f_\beta(\tau'))$$

where $f_\alpha(\tau) \in \frac{\text{diff}(S^1)^{\times M}}{SL(2,\mathbb{R})}$ – expanded reparametrization symmetry.
Each replica produces a soft mode with the total action

$$S_{local} \sim \beta J \int_0^{2\pi} \sum_\alpha \text{Sch}\left(\tan\left(f_\alpha(\theta)\right), \theta\right) \, d\theta$$

Results

- We constructed the new large N solutions of the SYK model
- The part of the free energy which is depend on replicas, has several metastable phases
- The factorized ansatz $G_{\alpha\beta}(\tau) = P_{\alpha\beta} g(\tau)$ gives a correction to the ground state energy $\beta F = \beta E_0 + \beta E_0^{RSB} - s_0 - \frac{c}{2\beta}$
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Thank you for your attention