$1/N$ diagrammatics of SYK model beyond the conformal limit

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based on work in progress with I. Aref'eva and M. Tikhanovskaya

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Quarks 2018

Sachdev-Ye-Kitaev (SYK) (Sachdev, Ye, '93; Kitaev, '15) model is a maximally chaotic (in the sense of the MSS bound (Maldacena, Shenker, Stanford, '15)) quantum mechanical many-body model that is solvable at large N.

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Applicability conditions of the holographic description are satisfied $\Rightarrow$ SYK is a solvable holographic model for 2D quantum gravity.

Definition of the model

Majorana fermions χ_i, $i = 1, \dots, N$, living in $(0+1)$ (Euclidean) dimensions

The Hamiltonian (Kitaev, '15):

$$H = \frac{1}{4!} \sum_{ijkl} j_{ijkl} \chi_i \chi_j \chi_k \chi_l$$

Random couplings j_{ijkl} - quenched disorder:

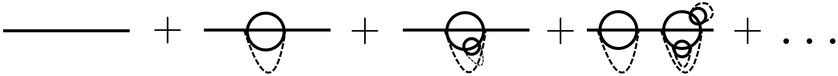
$$P(j_{ijkl}) = \sqrt{\frac{N^3}{12\pi J^2}}\, \mathrm{e}^{-\frac{N^3 j_{ijkl}^2}{12 J^2}}$$

$$\langle j_{ijkl} \rangle = 0\,, \qquad \langle j_{ijkl}^2 \rangle = \frac{3!\,J^2}{N^3}$$

Dimensionless coupling: βJ

Two-point function

$G(\tau_1, \tau_2) := \frac{1}{N} \sum_i \langle \chi_i(\tau_1) \chi_i(\tau_2) \rangle$ (Sachdev, Ye; Kitaev; Maldacena, Stanford)



Leading order in $1/N$ is captured by the melonic diagrams.

Self-energy: $\Sigma(\tau_1, \tau_2) = J^2 G(\tau_1, \tau_2)^3$.

Schwinger-Dyson equation for melonic graphs:

$$G(\omega) = \frac{1}{-i\omega - \Sigma(\omega)}$$

Conformal limit

IR limit: $\partial_\tau \to 0 \;\Leftrightarrow\; \beta J \gg 1$ (**strong coupling regime**)
In this case the large N SD equations in the position space reduce to

$$\int d\tau\, G(\tau_1,\tau)\Sigma(\tau,\tau_2) = -\delta(\tau_1-\tau_2)$$
$$\Sigma(\tau_1,\tau_2) = J^2 G(\tau_1,\tau_2)^3$$

The solution:

- For $\beta=\infty$: $G(\tau_1,\tau_2) \sim \frac{\text{sgn}(\tau_1-\tau_2)}{|J(\tau_1-\tau_2)|^{2\Delta}}$, where $\Delta = \frac{1}{4}$
- For finite β: $G(\tau_1,\tau_2) \sim \frac{\text{sgn}(\tau_1-\tau_2)}{\left|2\sin\frac{\pi}{\beta}(\tau_1-\tau_2)\right|^{2\Delta}}$

Reparametrization symmetry: $\tau \to f(\tau)$

$$G(\tau_1,\tau_2) \to f'(\tau_1)^\Delta f'(\tau_2)^\Delta G(f(\tau_1),f(\tau_2))$$
$$\Sigma(\tau_1,\tau_2) \to f'(\tau_1)^{1-\Delta} f'(\tau_2)^{1-\Delta} \Sigma(f(\tau_1),f(\tau_2))$$

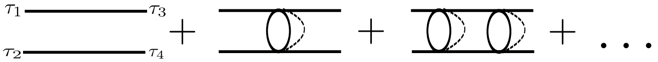
where $f \in \frac{\text{diff}(S^1)}{SL(2,\mathbb{R})}$ - plays the role of a soft Goldstone mode.

Four-point function

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\langle T(\chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4))\rangle = G(\tau_{12})G(\tau_{34}) + \frac{1}{N}\mathcal{F}(\tau_1,\tau_2,\tau_3,\tau_4) + \cdots$$

(Kitaev '15; Rosenhaus, Polchinski; Maldacena, Stanford '16)

$\mathcal{F}(\tau_1,\tau_2,\tau_3,\tau_4) =$



Define the (symmetric) ladder kernel:
$\tilde{K}(\tau_1,\tau_2,\tau_3,\tau_4) = -3J^2|G(\tau_1,\tau_2)|G(\tau_1,\tau_3)G(\tau_2,\tau_4)|G(\tau_1,\tau_4)|$
Then

$$\mathcal{F} \sim \sum_{n=0}^{\infty}\tilde{K}^n\mathcal{F}_0 = \frac{1}{1-\tilde{K}}\mathcal{F}_0 \sim \frac{\tilde{K}}{1-\tilde{K}}\cdot I$$

One solves the spectral problem for the ladder kernel. The result:

$$\mathcal{F} \sim \int_{\mathcal{C}} dh \frac{k(h)}{1-k(h)}\rho(h)\Psi_h(\tau_1,\tau_2;\tau_3,\tau_4)$$

Four-point function

In the conformal limit $\beta J \gg 1$, one obtains

$$\mathcal{F} = \mathcal{F}_{\text{non-conformal}} + \mathcal{F}_{\text{conformal}}$$

- Conformal part: $\mathcal{F}_{\text{conformal}} \sim \sum_{h_m} c_{\Delta,m} F_{\Delta,m}(1,2,3,4)$
  - conformal partial wave expansion in a 1D CFT
  - can be used to obtain conformal contributions to higher-point correlators (Gross, Rosenhaus, '17)
- Non-conformal part:

$$\mathcal{F}_{\text{non-conformal}} \sim \beta J \sum_n \text{Res}_{h=2} \left[\frac{k(h,n)}{1-k(h,n)} \rho(h) \right] \psi^*_{2,n}(\tau_1,\tau_2) \psi_{2,n}(\tau_3,\tau_4)$$

- comes from the soft mode corresponding to reparametrization symmetry
- responsible for maximal quantum chaos in SYK

Bilocal field path integral

- Average the partition function over the disorder:
  $Z = \int \mu[\vec{j}] d\vec{j} \int D\chi e^{-S_{SYK}[\chi, \vec{j}]}$
- Introduce the auxiliary bilocal fields $\tilde{\Sigma}$, $\tilde{G}$: $\tilde{\Sigma}(\tau_1, \tau_2)$ is a Lagrange multiplier which says that $\tilde{G}(\tau_1, \tau_2) = \frac{1}{N} \sum_i \chi_i(\tau_1) \chi_i(\tau_2)$
- Integrate out the fermions

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The result is (Kitaev; Maldacena, Stanford) $\quad Z = \int D\tilde{G} D\tilde{\Sigma}\ \mathrm{e}^{-S[\tilde{G},\tilde{\Sigma}]}$;

$$S[\tilde{G},\tilde{\Sigma}] = -\frac{N}{2}\mathrm{Tr}\log(\partial_\tau - \hat{\tilde{\Sigma}}) + \frac{N}{2}\int d\tau d\tau' \left(\tilde{\Sigma}(\tau,\tau')\tilde{G}(\tau,\tau') - \frac{J^2}{q}\tilde{G}(\tau,\tau')^q \right)$$

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- **Assumption:** the saddle point configuration is diagonal in replicas
  (see the talk by M. Tikhanovskaya for generalization)
- The saddle point is given by the large N Schwinger-Dyson equations
- Similar (but more complex) formalism to vector models
  (Aref'eva, '79)

Our motivation

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- The bilocal field path integral provides non-perturbative definition of a holographic quantum theory.
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- Our ultimate goal is to be able to identify the precise effects responsible for the information loss in the AdS_2 black hole using the SYK bilocal field path integral.

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- Our ultimate goal is to be able to identify the precise effects responsible for the information loss in the AdS_2 black hole using the SYK bilocal field path integral.
- We also would like to understand the full bulk theory more by studying the interactions of bulk matter fields with gravitational modes.

Main approach: systematic $1/N$ expansion

Suppose G, Σ - saddle point. Define new variables

$$g = |G|(\tilde{G} - G)\,, \quad \sigma = \frac{1}{J^2|G|}(\tilde{\Sigma} - \Sigma)$$

Semiclassical expansion in bilocal fields g, σ is the $1/N$ expansion of SYK.

Fermion correlators can be reconstructed as

$$\langle \chi_i \chi_i \rangle = \frac{1}{|G|}\langle g \rangle\,,\ \langle \chi_i \chi_i \chi_j \chi_j \rangle = \frac{1}{|G||G|}\langle gg \rangle\,,$$
$$\langle \chi_i \chi_i \chi_j \chi_j \chi_k \chi_k \rangle = \frac{1}{|G||G||G|}\langle ggg \rangle\,,\ \ldots$$

Quadratic action

(Maldacena, Stanford, '16)

$$S^{(2)} = -\frac{NJ^2}{12}\int d^4\tau\ \sigma(\tau_1,\tau_2)\tilde{K}(\tau_1,\tau_2,\tau_3,\tau_4)\sigma(\tau_3,\tau_4)$$
$$+\frac{NJ^2}{2}\int d\tau_1 d\tau_2\left[g(\tau_1,\tau_2)\sigma(\tau_1,\tau_2)-\frac{3}{2}g(\tau_1,\tau_2)^2\right].$$

Propagators:

$$\langle g(\tau_1,\tau_2)g(\tau_3,\tau_4)\rangle = \frac{2}{3NJ^2}\left(\frac{\tilde{K}}{1-\tilde{K}}\right)(\tau_1,\tau_2;\tau_3,\tau_4)\sim \mathcal{F}(\tau_1,\tau_2,\tau_3,\tau_4);$$
$$\langle g(\tau_1,\tau_2)\sigma(\tau_3,\tau_4)\rangle = \frac{2}{NJ^2}\left(\frac{1}{\tilde{K}-1}\right)(\tau_1,\tau_2;\tau_3,\tau_4);$$
$$\langle \sigma(\tau_1,\tau_2)\sigma(\tau_3,\tau_4)\rangle = \frac{6}{NJ^2}\left(\frac{1}{\tilde{K}-1}\right)(\tau_1,\tau_2;\tau_3,\tau_4).$$

Vertices

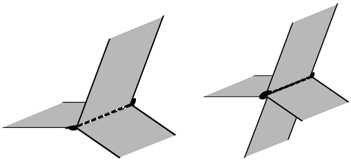
Contact vertices:

$$V_g^n := \frac{1}{n!}\frac{\delta^n S}{\delta g(\tau_1, \tau_{1'})\dots\delta g(\tau_n, \tau_{n'})}$$

Only two contact vertices:

$V_g^3 = -\frac{NJ^2}{2\ G(\tau_1,\tau_2)|G(\tau_1,\tau_2)|} g(\tau_1,\tau_2)^3$,

$V_g^4 = -\frac{NJ^2}{8\ G(\tau_1,\tau_2)^4} g(\tau_1,\tau_2)^4$

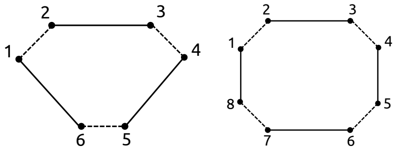


Planar vertices:

$$V_\sigma^n := \frac{1}{n!}\frac{\delta^n S}{\delta \sigma(\tau_1, \tau_{1'})\dots\delta \sigma(\tau_n, \tau_{n'})}$$

Infinite series of vertices is produced by the log term

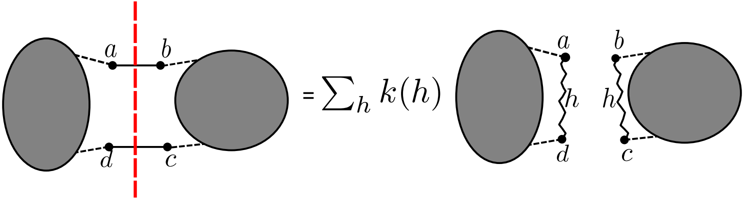
$\frac{N}{2}\log\left(1 - J^2\hat{G}\cdot|\hat{G}|\sigma\right)$



$n - th$ vertex is represented by a $2n$-gon.

Cutting rules for planar vertices

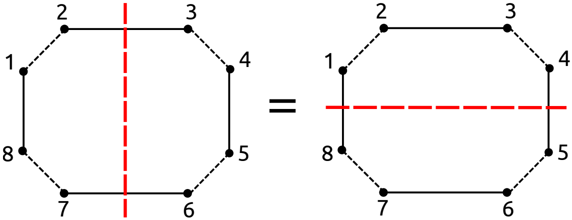
Suppose W is a tree-level planar amplitude, and $\Psi_{h,n}$ are (*exact*) eigenvectors of the ladder kernel.



$$W = U_1(\tau_1, \tau_a, \tau_d) G(\tau_a, \tau_b) G(\tau_c, \tau_d) U_2(\tau_2, \tau_b, \tau_c) =$$

$$= \sum_{h,n} k(h,n) U_1(\tau_1, \tau_a, \tau_d) \frac{\Psi^*_{h,n}(\tau_a, \tau_d)}{|G(\tau_a, \tau_d)|} \frac{\Psi_{h,n}(\tau_b, \tau_c)}{|G(\tau_b, \tau_c)|} U_2(\tau_2, \tau_b, \tau_c)$$

Eight-point function: cutting symmetry

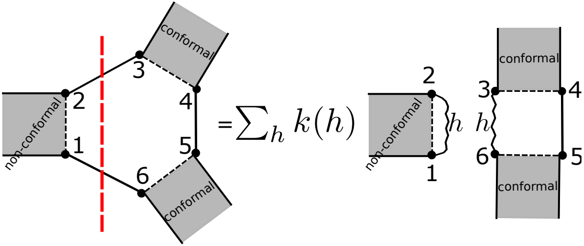


Different cuts of the 4-point vertex give the same result.
We can write a cutting symmetry equation for the integrands:

$$\frac{G(t_{81})}{|G(t_{27})|}\sum_{h,n} k(h,n)\Psi^*_{h,n}(t_2,t_7)\Psi_{h,n}(t_3,t_6)\frac{G(t_{45})}{|G(t_{36})|}$$

$$=\frac{G(t_{23})}{|G(t_{14})|}\sum_{h,n} k(h,n)\Psi^*_{h,n}(t_1,t_4)\Psi_{h,n}(t_8,t_5)\frac{G(t_{67})}{|G(t_{58})|}$$

Calculating non-conformal corrections to planar correlators



The prescription:

- Separate out the non-conformal contributions for every bilocal external line (no more than 1 cut per internal line)
- Expand the conformal external lines in terms of conformal eigenunctions
- Cut off all corners with non-conformal contributions: this reduces the number of integrals by 2 for every bilocal external line

Six-point function is reduced to a 3-loop integral, which can be computed.

Conclusions & outlook

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Thank you for attention!

Backup slides

Large N models

Model class

1) **Vector models**
   - exactly solvable at large N
   - no chaos at large N:
2) **Matrix models** (i.e. gauge theories).
   - (generally) not solvable even in the planar limit
   - can be maximally chaotic at strong coupling
3) **SYK-type models**
   - solvable at large N
   - maximally chaotic at strong coupling

Holographic dual

- Higher-spin gravity
  - *example*: 3D $O(N)$ vector theory $\leftrightarrow$ 4D HS theory
- Einstein gravity
  - *example*: Maldacena's AdS_5/CFT_4 gauge-gravity dualtity
- ?
  (gravitational sector - Jackiw-Teitelboim gravity)

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