# Modifying the theory of gravity by changing independent variables

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# Outline

- modified gravity $\Longrightarrow$ unified theory of gravity and electromagnetism;
- addition of matter:
  - ∗ classical point-like particle;
  - ∗ scalar field;
  - ∗ spinor field.

# Introduction

General Relativity - current description of gravity in modern physics

Consider modified gravity:

- presence of torsion: $S^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu} \neq 0$;
- metric and connection are independent variables (Palatini Formalism): $\Gamma^{\rho}_{\mu\nu} \neq \frac{1}{2} g^{\alpha\rho}(\partial_{\nu} g_{\alpha\mu} + \partial_{\mu} g_{\nu\alpha} - \partial_{\alpha} g_{\nu\mu})$.

# Gravity field equations without matter

The Einstein-Hilbert action:

$$S_1 = -\frac{1}{2\varkappa}\int d^4x\sqrt{-g}g^{\mu\nu}R_{\mu\nu}(\Gamma), \quad (1)$$

where $\varkappa$ is Einstein's constant, $R_{\mu\nu}$ is Ricci tensor, $g \equiv det g_{\mu\nu}$ .

How this action related to $\bar{S}_1 = -\frac{1}{2\varkappa}\int d^4x\sqrt{-g}g^{\mu\nu}\bar{R}_{\mu\nu}(\bar{\Gamma}(g))$?

$\bar{X} - Riemannian \;\; object.$

# Relation $S_1$ and $\bar{S}_1$

Introduce

$$\hat{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \bar{\Gamma}^{\alpha}_{\mu\nu} \longrightarrow R^{\mu}{}_{\nu\alpha\beta} = \bar{R}^{\mu}{}_{\nu\alpha\beta} + [\bar{D}_{\alpha}\hat{\Gamma}^{\mu}_{\beta\nu}]_{\alpha\beta} + [\hat{\Gamma}^{\mu}_{\alpha\xi}\hat{\Gamma}^{\xi}_{\beta\nu}]_{\alpha\beta}, \quad (2)$$

and

$$S_1 = -\frac{1}{2\varkappa}\int d^4x\sqrt{-g}(g^{\mu\nu}\bar{R}_{\mu\nu} + \hat{\Gamma}^{\mu}_{\mu\xi}\hat{\Gamma}^{\xi}_{\beta\alpha}g^{\beta\alpha} - \hat{\Gamma}^{\mu}_{\beta\xi}\hat{\Gamma}^{\xi}_{\mu\alpha}g^{\alpha\beta}) =$$

$$= \bar{S}_1 - \underbrace{\frac{1}{4\varkappa}\int d^4x\sqrt{-g}\hat{\Gamma}^{\mu}_{\alpha\gamma}\hat{\Gamma}^{\delta}_{\beta\nu}L_{\mu}{}^{\alpha\gamma,}{}_{\delta}{}^{\beta\nu}}_{\check{S}_1}, \quad (3)$$

where

$$L_{\mu}{}^{\alpha\gamma,}{}_{\delta}{}^{\beta\nu} = \delta^{\gamma}_{\delta}\delta^{\alpha}_{\mu}g^{\beta\nu} - \delta^{\gamma}_{\delta}\delta^{\beta}_{\mu}g^{\alpha\nu} + \delta^{\beta}_{\delta}\delta^{\nu}_{\mu}g^{\alpha\gamma} - \delta^{\alpha}_{\delta}\delta^{\nu}_{\mu}g^{\gamma\beta} \quad (4)$$

# Transformation of $\check{S}_1$

$$\check{S}_1 = \frac{1}{4\varkappa}\int d^4x\sqrt{-g}\,\hat{\Gamma}^{\mu}_{\alpha\gamma}\hat{\Gamma}^{\delta}_{\beta\nu}L_{\mu}{}^{\alpha\gamma,}{}_{\delta}{}^{\beta\nu} \tag{5}$$

Split $\hat{\Gamma}^{\alpha}_{\mu\nu}$:

$$\hat{\Gamma}^{\alpha}_{\mu\nu} = \underbrace{\tilde{\Gamma}^{\alpha}_{\mu\nu}}_{\tilde{\Gamma}^{\alpha}_{\mu\alpha}=0} + \underbrace{\frac{1}{4}\hat{B}_{\mu}\delta^{\alpha}_{\nu}}_{\hat{B}_{\mu}=\hat{\Gamma}^{\alpha}_{\mu\alpha}}. \tag{6}$$

$$\Longrightarrow \check{S}_1 = \tfrac{1}{4\varkappa}\int d^4x\sqrt{-g}\,\tilde{\Gamma}^{\mu}_{\alpha\gamma}\tilde{\Gamma}^{\delta}_{\beta\nu}L_{\mu}{}^{\alpha\gamma,}{}_{\delta}{}^{\beta\nu}$$

# Equations of motion

**Total action**

$$S_1 = \bar{S}_1 + \check{S}_1 = -\frac{1}{2\varkappa}\int d^4x\sqrt{-g}\,g^{\mu\nu}\bar{R}_{\mu\nu} - \frac{1}{4\varkappa}\int d^4x\sqrt{-g}\,\tilde{\Gamma}^{\mu}_{\alpha\gamma}\tilde{\Gamma}^{\delta}_{\beta\nu}L_{\mu}{}^{\alpha\gamma,}{}_{\delta}{}^{\beta\nu}$$

Independent variables: $g_{\mu\nu}; \Gamma^{\alpha}_{\mu\nu} \longrightarrow \tilde{\Gamma}^{\alpha}_{\mu\nu},\ \hat{B}_{\mu}$.

- $\delta\tilde{\Gamma}^{\alpha}_{\mu\nu} : \tilde{\Gamma}^{\alpha}_{\mu\nu} = 0$;
- $\delta\hat{B}_{\mu}$: $\hat{B}_{\mu}$ is missing out from action;
- $\delta g_{\mu\nu}$: $\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}g_{\mu\nu} = 0$.

Therefore, full equivalence to the GR.

# Field $\hat{B}_\mu$

Consider $R^{\mu}{}_{\mu\alpha\beta}$:

$$R^{\mu}{}_{\mu\alpha\beta} = [\partial_\alpha \Gamma^{\mu}_{\beta\mu}]_{\alpha\beta} = [\partial_\alpha \hat{\Gamma}^{\mu}_{\beta\mu}]_{\alpha\beta} = \partial_\alpha \hat{B}_\beta - \partial_\beta \hat{B}_\alpha. \tag{7}$$

Construct an additional action:

$$S_2 = -\theta \int d^4x \sqrt{-g} R^{\mu}{}_{\mu\alpha\beta} R^{\nu}{}_{\nu\rho\sigma} g^{\alpha\rho} g^{\beta\sigma}, \tag{8}$$

So, $\delta\hat{B}$: $\bar{D}_\alpha(\partial_\mu \hat{B}_\nu - \partial_\nu \hat{B}_\mu) = 0$.
Under U(1) transformation: $\hat{B}_\mu \longmapsto \hat{B}_\mu - \partial_\mu \alpha$.
$\hat{B}$ - like an electromagnetic potential!

# Point-like particle

$$S_m + S_3 = -m \int d\tau \sqrt{\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) g_{\mu\nu}(x(\tau))} - q \int d\tau \dot{x}^\mu(\tau) \hat{B}_\mu(x(\tau)), \tag{9}$$

## Equations of motion

$$\begin{cases} \tilde{\Gamma}^\alpha_{\mu\nu} = 0; \\ \bar{R}_{\mu\nu} - \frac{1}{2}\bar{R} g_{\mu\nu} = \varkappa (T_{1\mu\nu} + T_{2\mu\nu}); \qquad \longrightarrow gravity \\ \bar{D}_\mu F^{\mu\nu} = 4\pi j^\nu; \qquad +electrodynamic \\ m u^\mu \bar{D}_\mu u^\alpha = -q u_\xi F^{\xi\alpha}, \end{cases}$$

where $F_{\mu\nu} \equiv R^\alpha{}_{\alpha\mu\nu}$, $\qquad j^\nu = q \int ds u^\nu \delta(x - x(s)) \frac{1}{\sqrt{-g}}$,

$$T_{1\mu\nu} + T_{2\mu\nu} = \rho_m u_\mu u_\nu - \frac{1}{4\pi}(F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}),$$

$$\rho_m = m \int ds \delta(x - x(s)) \frac{1}{\sqrt{-g}}, \qquad u^\mu = \dot{x}^\mu \frac{1}{\sqrt{\dot{x}^\alpha \dot{x}^\beta g_{\alpha\beta}}}.$$

# Matter field

Scalar field $\varphi$:

Instead of $S_m + S_3 \longrightarrow S_\varphi$:

$$S_\varphi = \int d^4x \sqrt{-g}((\partial_\mu + i\hat{B}_\mu)\varphi)^*(\partial_\nu + i\hat{B}_\mu)\varphi)g^{\mu\nu}. \tag{10}$$

Total action:

$$S = -\frac{1}{2\varkappa}\int d^4x\sqrt{-g}\,g^{\mu\nu}\bar{R}_{\mu\nu} - \frac{1}{4\varkappa}\int d^4x\sqrt{-g}\tilde{\Gamma}^\mu_{\alpha\gamma}\tilde{\Gamma}^\delta_{\beta\nu}L_\mu{}^{\alpha\gamma,}{}_\delta{}^{\beta\nu} - \\ - \theta\int d^4x\sqrt{-g}R^\mu{}_{\mu\alpha\beta}R^\nu{}_{\nu\rho\sigma}g^{\alpha\rho}g^{\beta\sigma} + S_\varphi \tag{11}$$

# Dirac spinor field $\Psi$

Metric $g_{\mu\nu} \longrightarrow$ tetrad $e^a_\mu$: $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$;

$$\bar{D}_\mu e^a_\nu = \partial_\mu e^a_\nu - \bar{\Gamma}^\alpha_{\mu\nu} e^a_\alpha + \bar{\omega}_\mu{}^a{}_b e^b_\nu = 0 \longrightarrow \bar{\omega}_\mu{}^a{}_b = \bar{\Gamma}^\alpha_{\mu\nu} e^a_\alpha e^\nu_b - e^\nu_b \partial_\mu e^a_\nu .$$

$$S_\Psi = \int d^4x \sqrt{-g}\, \bar{\Psi} \gamma^c e^\mu_c (\partial_\mu + \bar{\omega}_{\mu ab}[\gamma^a, \gamma^b] + i\hat{B}_\mu)\Psi, \qquad (12)$$

Total action:

$$S = -\frac{1}{2\varkappa}\int d^4x \sqrt{-g}\, g^{\mu\nu}\bar{R}_{\mu\nu} - \frac{1}{4\varkappa}\int d^4x \sqrt{-g}\, \tilde{\Gamma}^\mu_{\alpha\gamma}\tilde{\Gamma}^\delta_{\beta\nu} L_\mu{}^{\alpha\gamma,}{}_\delta{}^{\beta\nu} - \\ - \theta \int d^4x \sqrt{-g}\, R^\mu{}_{\mu\alpha\beta} R^\nu{}_{\nu\rho\sigma} g^{\alpha\rho} g^{\beta\sigma} + S_\Psi \qquad (13)$$

# Generalization to $GL(2, C)$

Whel spinor $\chi^A$ and $(\chi^A)^* \equiv \chi^{\dot{A}}$:

$$e_a^\mu \longrightarrow e^\mu_{A\dot{B}} = \frac{1}{\sqrt{2}} \sigma^a_{A\dot{B}} e_a^\mu; \tag{14}$$

$$(e^\mu_{A\dot{B}})^* = e^\mu_{B\dot{A}}. \tag{15}$$

$$S_\chi = \int d^4x \sqrt{-g} (i\chi^{\dot{A}} e^\mu_{B\dot{A}} D_\mu \chi^B - i\chi^A e^\mu_{A\dot{B}} D_\mu \chi^{\dot{B}}), \tag{16}$$

where $D_\mu \chi^B = (\delta^A_C \partial_\mu + \omega_\mu{}^A{}_C)\chi^C$.

## Constructing metric

$$\varepsilon_{AB} \longrightarrow E_{AB}(x) = \varphi(x) \varepsilon_{AB}, \tag{17}$$

$$g_{\mu\nu} = e^{A\dot{B}}_\mu e^{C\dot{D}}_\nu \varepsilon_{AC} \varepsilon_{\dot{B}\dot{D}} \longrightarrow g_{\mu\nu} = e^{A\dot{B}}_\mu e^{C\dot{D}}_\nu E_{AC} E_{\dot{B}\dot{D}}. \tag{18}$$

# Generalization to GL(2,C)

$\Gamma^{\alpha}_{\mu\nu}$ is not independent variable:

$$D_\mu e_\nu^{A\dot{B}} = \partial_\mu e_\nu^{A\dot{B}} - \Gamma^{\sigma}_{\mu\nu} e_\sigma^{A\dot{B}} + {\omega_\mu}^{A}{}_{C} e_\nu^{C\dot{B}} + {\omega_\mu}^{\dot{B}}{}_{\dot{D}} e_\nu^{A\dot{D}} = 0 \tag{19}$$

$$\longrightarrow \Gamma^{\alpha}_{\mu\nu} = e^{\alpha}_{A\dot{B}}(\partial_\mu e_\nu^{A\dot{B}} + {\omega_\mu}^{A}{}_{C} e_\nu^{C\dot{B}} + {\omega_\mu}^{\dot{B}}{}_{\dot{D}} e_\nu^{A\dot{D}}) \tag{20}$$

Split connection:

$$ {\omega_\mu}^{A}{}_{B} = \underbrace{{\tilde{\omega}_\mu}{}^{A}{}_{B}}_{{\tilde{\omega}_\mu}{}^{A}{}_{A}=0} + \frac{1}{2}(iA_\mu + B_\mu)\delta^A_B \tag{21}$$

Change variables

$$e_\mu^{A\dot{B}} \longrightarrow \tilde{e}_\mu^{A\dot{B}} = e_\mu^{A\dot{B}}|\varphi|; \quad \chi^A \longrightarrow \tilde{\chi}^A = \chi^A\sqrt{|\varphi|}$$

Introduce $\hat{\omega}_\mu{}^A{}_B$:

$$\hat{\omega}_\mu{}^A{}_B = \omega_\mu{}^A{}_B - \bar{\omega}_\mu{}^A{}_B \tag{22}$$

Define Riemannian connection $\bar{\omega}_\mu{}^A{}_B$:

$$\partial_\mu \tilde{e}_\nu^{A\dot{B}} - \bar{\Gamma}^\alpha_{\mu\nu}\tilde{e}_\alpha^{A\dot{B}} + \bar{\omega}_\mu{}^{\dot{A}}{}_{\dot{C}}\tilde{e}_\nu^{B\dot{C}} + \bar{\omega}_\mu{}^B{}_D\tilde{e}_\nu^{D\dot{A}} = 0 \tag{23}$$

So, rewrite connection:

$$\omega_\mu{}^A{}_B = \bar{\omega}_\mu{}^A{}_B + \hat{\omega}_\mu{}^A{}_B + \frac{1}{2}(iA_\mu + B_\mu)\delta^A_B; \tag{24}$$

$$\tilde{\Gamma}_{\alpha\mu\nu} = 2Re(\hat{\omega}_{\mu BD}\tilde{e}_\alpha^{B\dot{C}}\tilde{e}_\nu^{D\dot{A}}E_{\dot{A}\dot{C}}). \tag{25}$$

Independent variables: $\tilde{e}_\mu^{A\dot{B}}$; $\tilde{\chi}^A$; $\omega_\mu{}^A{}_B \longrightarrow \tilde{\Gamma}_{\alpha\mu\nu}, B_\mu, A_\mu$

Instead of $-\theta \int d^4x\sqrt{-g}R^{\mu}{}_{\mu\alpha\beta}R^{\nu}{}_{\nu\rho\sigma}g^{\alpha\rho}g^{\beta\sigma}$:

$$S_2 = -\theta \int d^4x\sqrt{-g}g^{\alpha\mu}g^{\beta\nu}F^{A}{}_{A\alpha\beta}F^{B}{}_{B\mu\nu}, \tag{26}$$

where $F^{A}{}_{B\mu\nu} = [\partial_\mu \omega_\nu{}^{A}{}_{B} + \omega_\mu{}^{A}{}_{C}\omega_\nu{}^{C}{}_{B}]_{\mu\nu}$

Total action

$$S = -\frac{1}{2\varkappa}\int d^4x\sqrt{-g}g^{\mu\nu}\bar{R}_{\mu\nu} -$$
$$-\theta \int d^4x\sqrt{-g}g^{\alpha\rho}g^{\beta\sigma}((\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) +$$
$$+ (\partial_\alpha B_\beta - \partial_\beta B_\alpha)(\partial_\rho B_\sigma - \partial_\sigma B_\rho)) +$$
$$+ \int d^4x\sqrt{-g}(\tilde{\chi}^{\dot{A}} i\tilde{e}^{\mu}_{B\dot{A}}(\delta^B_C\partial_\mu + \bar{\omega}_\mu{}^{B}{}_{C} + iA_\mu\delta^B_C)\tilde{\chi}^C + h.c.) -$$
$$-\frac{1}{4\varkappa}\int d^4x\sqrt{-g}\tilde{\Gamma}_{\mu\alpha\gamma}\tilde{\Gamma}_{\delta\beta\nu}L^{\mu\alpha\gamma,\delta\beta\nu} + \frac{1}{\sqrt{8}}\int d^4x\sqrt{-g}E_{\beta\alpha\nu\mu}\tilde{\Gamma}^{\alpha\nu\mu}\tilde{\chi}^{\dot{A}}\tilde{e}^{\beta}_{B\dot{A}}\tilde{\chi}^B$$

## Variation $\delta\tilde{\Gamma}$:

$$\tilde{\Gamma}_{\mu\alpha\gamma} = CL_{\mu\alpha\gamma,\delta\beta\nu}E^{\sigma\delta\beta\nu}j_{\sigma} \Longrightarrow$$

$$S_{ad} = \frac{1}{4\varkappa}\int d^4x\sqrt{-g}\tilde{\Gamma}_{\mu\alpha\gamma}\tilde{\Gamma}_{\delta\beta\nu}L^{\mu\alpha\gamma,\delta\beta\nu} + \\ + \frac{1}{\sqrt{8}}\int d^4x\sqrt{-g}E_{\beta\alpha\nu\mu}\tilde{\Gamma}^{\alpha\nu\mu}\underbrace{\tilde{\chi}^{\dot{A}}\tilde{e}^{\beta}_{B\dot{A}}\tilde{\chi}^{B}}_{j^{\beta}} = \\ = \beta\int d^4x\sqrt{-g}j^{\alpha}j^{\beta}g_{\alpha\beta}, \quad (27)$$

where

$$j^{\alpha}j^{\beta}g_{\alpha\beta} = \tilde{\chi}^{\dot{A}}\tilde{e}^{\alpha}_{B\dot{A}}\tilde{\chi}^{B}\tilde{\chi}^{\dot{C}}\tilde{e}^{\beta}_{D\dot{C}}\tilde{\chi}^{D}g_{\alpha\beta} = \tilde{\chi}^{\dot{A}}\tilde{\chi}^{B}\tilde{\chi}^{\dot{C}}\tilde{\chi}^{D}\varepsilon_{BD}\varepsilon_{\dot{A}\dot{C}} = 0 \quad (28)$$

Final action:

$$S = -\frac{1}{2\varkappa}\int d^4x\sqrt{-g}g^{\mu\nu}\bar{R}_{\mu\nu} -$$
$$-\theta\int d^4x\sqrt{-g}g^{\alpha\rho}g^{\beta\sigma}((\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) +$$
$$+(\partial_\alpha B_\beta - \partial_\beta B_\alpha)(\partial_\rho B_\sigma - \partial_\sigma B_\rho)) +$$
$$+\int d^4x\sqrt{-g}(\tilde{\chi}^{\dot{A}} i\tilde{e}^{\mu}_{B\dot{A}}(\delta^B_C\partial_\mu + \bar{\omega}_\mu{}^B{}_C + iA_\mu\delta^B_C)\tilde{\chi}^C + h.c.) \quad (29)$$

Therefore, this is a standart action to the Einstain-Dirac-Maxwell theory. Electromagnetic potential $A_\mu$ is a component of imaginary trace of connection.

# Properties of $L_{\mu}{}^{\alpha\gamma,}{}_{\delta}{}^{\beta\nu}$

$$L_{\mu}{}^{\alpha\gamma,}{}_{\delta}{}^{\beta\nu} = \delta^{\gamma}_{\delta}\delta^{\alpha}_{\mu}g^{\beta\nu} - \delta^{\gamma}_{\delta}\delta^{\beta}_{\mu}g^{\alpha\nu} + \delta^{\beta}_{\delta}\delta^{\nu}_{\mu}g^{\alpha\gamma} - \delta^{\alpha}_{\delta}\delta^{\nu}_{\mu}g^{\gamma\beta}$$

- $64 \times 64$ matrix;
- symmetric: $L^{IK} = L^{KI}$, $I \equiv \{_{\mu}{}^{\alpha\gamma}\}$;
- 4-dimensional zero proper subspace: $L_{\mu}{}^{\alpha\mu,}{}_{\delta}{}^{\beta\nu} = 0$;
- $\longrightarrow$ rank 60.