Parity anomaly in four dimensions

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by M.K. and Dmitri Vassilevich

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Outline:

* Parity anomaly in 3D: a brief review.
* Confined fermions in 4D.
* Parity anomaly in 4D: the definition.
* Parity anomaly in 4D: the computation.
* Parity anomaly in 4D: the gravitational contribution.
* Summary.

Parity anomaly in 3D

- Niemi, Semenoff - Phys. Rev. Lett. 51, 2077 (1983)
  Redlich - Phys. Rev. Lett. 52, 18 (1984)

  QED of massless fermions in 3D.

- At the **classical** level the massless fermionic field $\psi(x)$ satisfies:

$$\left(\not{D}[A]\psi\right)(x) = 0, \quad \not{D}[A] = \mathrm{i}\gamma^i\left(\frac{\partial}{\partial x^i} + \mathrm{i}A_i(x)\right)$$

- This e.o.m. is invariant upon the parity transformation:

$$\left.\begin{array}{l}\psi(x) \longrightarrow \psi(-x) \\ A_i \longrightarrow -A_i(-x)\end{array}\right\} \Rightarrow \not{D}[A]\psi(x) \longrightarrow -\left(\not{D}[A]\psi\right)(-x) = 0$$

Parity anomaly in 3D

- Quantization:

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{-\int d^3x \, \bar{\psi} \not{D}[A] \psi} = \text{det} \left(\not{D}[A] \right) = e^{-W[A]}$$

$$W[A] = -\log \det \not{D} \quad - \quad \text{the one loop effective action}$$

- The one-loop radiative correction $W[A]$ breaks the parity symmetry $A_i(x) \longrightarrow -A_i(-x)$ due to the Chern-Simons term!

$$W[A] = \pm \frac{\mathrm{i}}{8\pi} \int d^3x \; A_i \partial_j A_k \epsilon^{ijk} + \text{ parity-even terms}$$

Parity anomaly in 3D

- Let us track the origin of this anomaly.

$$e^{-W[A]} = \det\left(\not{D}[A]\right) = \prod_{\lambda \in \mathrm{Spec}(\not{D}[A])} \lambda$$

- The parity transformation reflects the spectrum of the Dirac operator:

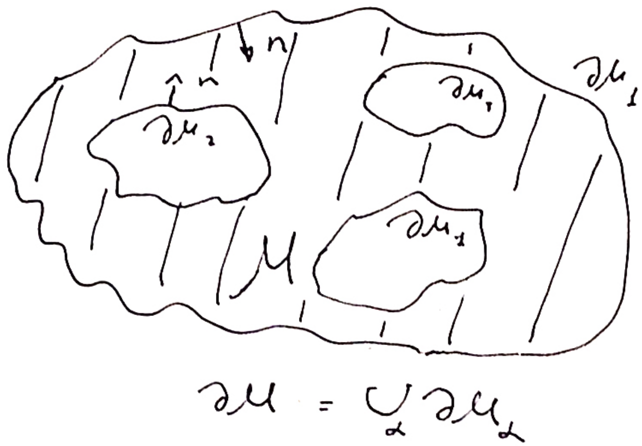
$$\left(\not{D}[A]\psi\right)(x) = \lambda\psi(x) \longrightarrow \left(-\not{D}[A]\psi\right)(-x) = \lambda\psi(-x)$$

- Parity anomaly expresses a spectral asymmetry of the Dirac operator:

$$W_{\mathrm{odd}} = -\frac{1}{2}\left(\log\det\left(\not{D}[A]\right) - \log\det\left(-\not{D}[A]\right)\right) \neq 0$$

Confined fermions in 4D: general setup

We are dealing with the Euclidean 4D manifold $\mathcal{M}$ with a boundary $\partial\mathcal{M}$



The Dirac operator is the usual one

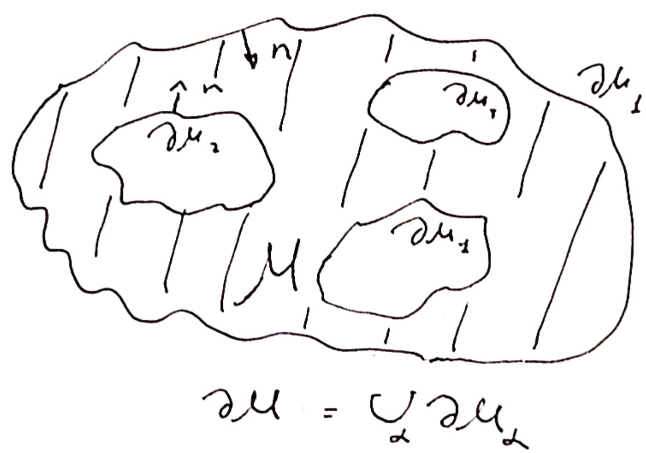
$$\begin{aligned} \not{D} &= \mathrm{i}\gamma^\mu (\nabla_\mu + \mathrm{i}A_\mu) \\ \nabla_\mu &= \partial_\mu + \omega_\mu \end{aligned}$$

Boundary conditions must satisfy:

- $\not{D}^\dagger = \not{D}$
- $\left(\bar{\psi}\gamma^\mu\psi\right) n_\mu \Big|_{\partial\mathcal{M}} = 0$

Confined fermions in 4D: the boundary conditions

We exploit the (Euclidean version of) the MIT bag boundary conditions



For each component of the boundary $\partial\mathcal{M}_\alpha$ we define the projectors

$$\Pi_\pm = \frac{1}{2}\left(1 \pm \mathrm{i}\epsilon_\alpha \gamma^5 \gamma^n\right), \quad \epsilon_\alpha \in \{-1, +1\}$$

and impose the boundary conditions

$$\Pi_- \psi\Big|_{\partial\mathcal{M}} = 0$$

These boundary conditions guarantee that:

- $\not{D}^\dagger = \not{D}$
- $\left(\bar{\psi}\gamma^\mu\psi\right) n_\mu\Big|_{\partial\mathcal{M}} = 0$

Parity anomaly in 4D: the definition

The parity anomaly can be defined in arbitrary dimension via

$$W_{\text{odd}} = -\frac{1}{2}\left(\log\det\left(\not{D}[A]\right) - \log\det\left(-\not{D}[A]\right)\right)$$

however this expression does not make any sense, unless one introduces the regularisation.

ζ-function regularisation of the fermionic determinant:

$$W\left[\not{D}\right] \equiv -\log\det\left(\not{D}[A]\right) \longrightarrow \mu^s\Gamma(s)\zeta(s,\not{D}) \equiv W_s[\not{D}], \qquad s \to 0$$

where the zeta function is defined in the following way:

$$\zeta(s,\not{D}) = \sum_{\lambda>0}\lambda^{-s} + e^{-\mathrm{i}\pi s}\sum_{\lambda<0}(-\lambda)^{-s}$$

Parity anomaly in 4D: the definition

We are interested in the parity-odd contribution

$$\zeta_{\text{odd}}\left(s,\not{D}\right) \equiv \frac{1}{2}\left(\zeta_{\text{odd}}\left(s,\not{D}\right) - \zeta_{\text{odd}}\left(s,-\not{D}\right)\right) = \frac{1}{2}\left(1 - e^{-\mathrm{i}\pi s}\right)\eta(s,\not{D}),$$

where

$$\eta\left(s,\not{D}\right) = \sum_{\lambda>0}\lambda^{-s} - \sum_{\lambda<0}(-\lambda)^{-s}.$$

therefore (L. Alvarez-Gaume, S. Della Pietra and G.Moore, 1984)

$$W_{\text{odd}} = \lim_{s\to 0}\mu^s\Gamma(s)\zeta_{\text{odd}}(s,\not{D}) = \frac{\mathrm{i}\pi}{2}\eta(0,\not{D}) = \text{finite!}$$

Why do we expect it to be different from zero???

- When there is no boundary the spectrum is symmetric with respect to 0, since $\left\{\not{D},\gamma^5\right\} = 0$.
- Boundary conditions break this property: $\Pi_-\gamma^5 = \gamma^5\Pi_+$

Parity anomaly in 4D: the computation.

The spectral function $\eta(s, \not{D})$ exhibits the following integral representation:

$$\eta(s, \not{D}) = \frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty \tau^s \mathrm{Tr}\left(\not{D} e^{-\tau^2 \not{D}^2}\right)$$

Let us consider the variation of the gauge field $A_\mu(x) \longrightarrow A_\mu(x) + \delta A_\mu(x)$:

$$\delta\eta(s, \not{D}) = \frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty d\tau\, \tau^s \frac{d}{d\tau} \mathrm{Tr}\ \underbrace{\left(\delta \not{D}\right)}_{-\gamma^\mu \delta A_\mu}\ \tau e^{-\tau^2 \not{D}^2}.$$

At the physical limit $s = 0$

$$\delta\eta(0, \not{D}) = -\frac{2}{\sqrt{\pi}} \lim_{t \longrightarrow +0} \mathrm{Tr}\left(\delta \not{D}\right) \sqrt{t} e^{-t \not{D}^2}$$

Parity anomaly in 4D: the computation.

For an arbitrary matrix-valued function Q the following asymptotic expansion holds at $t \longrightarrow +0$:

$$\mathrm{Tr}\, Q\, e^{-t\not{D}^2} \simeq \sum_{k=0}^{\infty} t^{\frac{k-4}{2}} a_k \left(Q, \not{D}^2\right)$$

The relevant heat kernel coefficients were studied by V. N. Marachevsky and D. V. Vassilevich in Nucl. Phys. B677, 535 (2004)

In our case a_0, a_1 and a_2 vanish, therefore

$$\begin{aligned} \delta\eta(0, \not{D}) &= -\frac{2}{\sqrt{\pi}} a_3 \left(\delta \not{D}, \not{D}^2\right) \\ &= \left(-\frac{1}{4\pi^2}\right) \sum_{\alpha} \int_{\partial \mathcal{M}_\alpha} d^3x \sqrt{h}\, \epsilon_\alpha\, \varepsilon^{nabc} \left(\delta A_a\right) \partial_b A_c \end{aligned}$$

Parity anomaly in 4D: the computation.

The variation of the parity-odd contribution to the one-loop effective action reads:

$$\delta W_{\text{odd}} = \left(-\frac{\mathrm{i}}{8\pi}\right) \sum_\alpha \int_{\partial\mathcal{M}_\alpha} d^3x \sqrt{h}\, \epsilon_\alpha\, \varepsilon^{nabc}\, (\delta A_a)\, \partial_b A_c.$$

In terms of the induced current the answer reads:

$$\mathcal{J}^a_{\text{odd}}(x) \equiv \frac{1}{\sqrt{h}} \frac{\delta W_{\text{odd}}}{\delta A_a(x)} = -\frac{\mathrm{i}}{8\pi}\, \epsilon_\alpha(x) \varepsilon^{nabc} \partial_b A_c = -\frac{\mathrm{i}}{16\pi}\, \epsilon_\alpha(x) \varepsilon^{nabc} F_{bc}$$

If the gauge potential A is defined globally on $\mathcal{M}$, we can recover W_{odd}

$$W_{\text{odd}} = \frac{1}{4} \sum_\alpha \epsilon_\alpha \left(-\frac{\mathrm{i}}{4\pi} \int_{\partial\mathcal{M}_\alpha} \sqrt{h} \varepsilon^{nabc} A_a \partial_b A_c\right)$$

Comment on the gauge invariance of the answer.

Let us rewrite the answer in terms of the differential forms. We assume that

$$A = A_\mu \, dx^\mu \in D^1 \left(\mathcal{M} \cup \partial \mathcal{M} \right).$$

Then the answer reads:

$$\begin{aligned} W_{\mathrm{odd}} &= \mathrm{i} \sum_\alpha \mathrm{CS}_\alpha \left[k_\alpha, A \right], \\ \mathrm{CS}_\alpha \left[k_\alpha, A \right] &= \frac{k_\alpha}{4\pi} \int_{\partial \mathcal{M}_\alpha} A \wedge dA, \quad k_\alpha = \varepsilon_\alpha \cdot \frac{1}{4}. \end{aligned}$$

Each contribution is invariant upon the gauge transformation:

$$A \longrightarrow A - \mathrm{i} U^{-1} dU, \quad \forall U \in C^\infty(\mathcal{M} \cup \partial \mathcal{M}), \quad |U| = 1.$$

Indeed

$$\begin{aligned} d(U^{-1} dU) &= 0 \quad \Rightarrow \quad dA \longrightarrow dA, \\ A \wedge dA &\rightarrow A \wedge dA - \mathrm{i} U^{-1} dU \wedge dA = A \wedge dA - \mathrm{i} d \left(U^{-1} dU \wedge A \right) \\ \int_{\partial \mathcal{M}_\alpha} A \wedge dA &\rightarrow \int_{\partial \mathcal{M}_\alpha} A \wedge dA - \mathrm{i} \underbrace{\int_{\partial \mathcal{M}_\alpha} d \left(U^{-1} dU \wedge A \right)}_{0} = \int_{\partial \mathcal{M}_\alpha} A \wedge dA. \end{aligned}$$

Comment on the classical symmetry.

Let us consider the transformations which leave invariant the classical e.o.m.

$$\begin{cases} \mathrm{i}\gamma^\mu\left(\frac{\partial}{\partial x^\mu}+\mathrm{i}A_\mu(x)\right)\psi=0 \\ \frac{1}{2}\left(\mathbf{1}-\mathrm{i}\epsilon_\alpha\gamma^5\gamma^n\right)\psi\Big|_{\partial\mathcal{M}}=0 \end{cases}, \quad \epsilon_\alpha\in\{+1,-1\}.$$

The transformation

$$\begin{cases} \psi(x)\longrightarrow\psi(-x) \\ A_\mu(x)\longrightarrow -A_\mu(-x) \end{cases}$$

is a bad candidate: if $x\in\mathcal{M}$ there is no guarantee that $-x\in\mathcal{M}$.

The correct classical symmetry in a presence of the boundary is

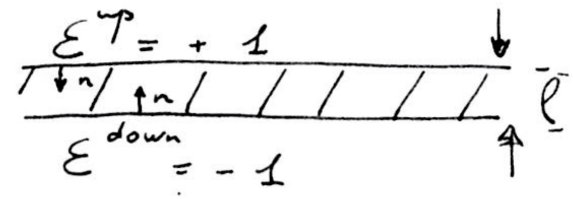
$$\begin{cases} \psi(x)\longrightarrow\gamma^5\psi(x) \\ \epsilon_\alpha\longrightarrow-\epsilon_\alpha \end{cases}.$$

This transformation inverts the nonzero spectrum of the Dirac operator.

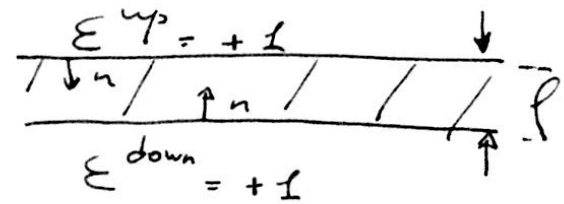
Comparison: 3D v.s. 4D results

4D case: $\quad |k| = \frac{1}{4}$

3D case: $\quad |k| = \frac{1}{2}$



Why is it natural?



Let us consider $\mathcal{M} = \mathbb{R}^3 \times [0; l]$ at $l \to 0$.

at $\quad \epsilon^{\text{up}} = -\epsilon^{\text{down}} \quad \text{CS}[1/4, A] - \text{CS}[-1/4, A] = \text{CS}[1/2, A] \Rightarrow$ 3D-result,

at $\quad \epsilon^{\text{up}} = +\epsilon^{\text{down}} \quad \text{CS}[1/4, A] - \text{CS}[+1/4, A] = 0 \Rightarrow$ nothing

Let us take a look at the spectrums of massless $\not{D}$ in both cases at $A = 0$

- at $\epsilon^{\text{up}} = -\epsilon^{\text{down}}$ we obtain $\lambda^2(p, k_{||}) = k_{||}^2 + m_p^2$, $m_p^2 = \frac{\pi p^2}{l^2}$, $p \in \mathbb{Z}$. At $l \to 0$ massless modes with $p = 0$ survive $\longrightarrow$ massless 3D spectrum.

- at $\epsilon^{\text{up}} = +\epsilon^{\text{down}}$ we obtain $\lambda^2(p, k_{||}) = k_{||}^2 + m_p^2$, $m_p^2 = \frac{\pi\left(p+\frac{1}{2}\right)^2}{l^2}$, $p \in \mathbb{Z}$. At $l \to 0$ all eigenstates become infinitely massive.

Gravitational contribution: the 3D case.

- What about the gravitational contribution to the parity anomaly?

$$\begin{aligned} \not{D} &= \mathrm{i}\gamma^i \nabla_i, \quad \nabla_i = \partial_i + \omega_i \\ W_{\text{odd}} &= -\left(\log\det\left(\not{D}\right) - \log\det\left(-\not{D}\right)\right) \neq 0? \end{aligned}$$

- The answer is "yes", it is different from zero.

$$W_{\text{odd}} = -\frac{\mathrm{i}k}{4\pi}\int d^3x\sqrt{g}\epsilon^{\mu\nu\rho}\left(\Gamma^{\lambda}_{\mu\kappa}\partial_\nu\Gamma^{\kappa}_{\rho\lambda} + \frac{2}{3}\Gamma^{\lambda}_{\mu\kappa}\Gamma^{\kappa}_{\nu\sigma}\Gamma^{\sigma}_{\rho\lambda}\right).$$

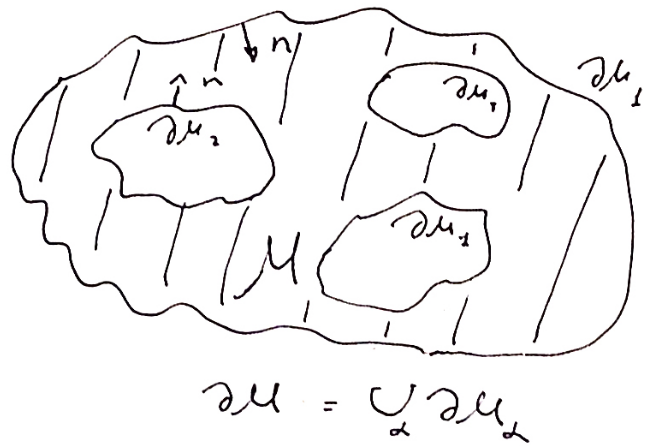
- There have been contradicting results in the literature regarding the coefficient k in front of Chern-Simons term.

$k = \frac{1}{48}$, (Goni, Valle -1986; Vuorio -1986; van der Bij - 1986)

$k = \frac{1}{16}$, (Ojima -1989)

Gravitational contribution to the 4D parity-anomaly.

We are dealing with the Euclidean 4D manifold $\mathcal{M}$ with a boundary $\partial\mathcal{M}$



The Dirac operator is the usual one with the bag boundary conditions:

$$\begin{cases} \not{D} = \mathrm{i}\gamma^{\mu}\left(\partial_{\mu} + \omega_{\mu}\right) \\ \frac{1}{2}\left(1 - \mathrm{i}\epsilon_{\alpha}\gamma^{5}\gamma^{n}\right)\psi\Big|_{\partial\mathcal{M}} = 0 \end{cases}, \quad \epsilon_{\alpha} \in \{+1, -1\}.$$

Upon the zeta-function regularization the parity anomaly reads:

$$W_{\mathrm{odd}} = \frac{\mathrm{i}\pi}{2}\eta(0, \not{D}),$$

where $\eta\left(s, \not{D}\right) = \sum_{\lambda>0} \lambda^{-s} - \sum_{\lambda<0} (-\lambda)^{-s}.$

Parity anomaly in 4D: the computation.

Using again the integral representation:

$$\eta(s, \not{D}) = \frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty \tau^s \mathrm{Tr}\left(\not{D} e^{-\tau^2 \not{D}^2}\right)$$

Let us consider the variation of the vierbeins $e_{\mu a} \longrightarrow e_{\mu a} + \delta e_{\mu a}$:

$$\begin{aligned} \delta\eta(s, \not{D}) &= \frac{2}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^\infty d\tau\, \tau^s \frac{d}{d\tau} \mathrm{Tr}\, (\delta\not{D}) \tau e^{-\tau^2 \not{D}^2}, \\ \delta\not{D} &= \mathrm{i}\gamma^\mu \delta\omega_\mu + \mathrm{i}\, (\delta e_a^\mu)\, \gamma^a \nabla_\mu - \textbf{1-st order diff. operator!} \end{aligned}$$

At the physical limit $s = 0$

$$\delta\eta(0, \not{D}) = -\frac{2}{\sqrt{\pi}} \lim_{t \longrightarrow +0} \mathrm{Tr}\left(\delta\not{D}\right) \sqrt{t} e^{-t\not{D}^2}$$

Parity anomaly in 4D: the computation.

For the first order diff op. $Q = Q_1^\mu \partial_\mu + Q_0$ the asymptotic expansion at $t \longrightarrow +0$ has a different structure:

$$\mathrm{Tr}\, Q\, e^{-t\not{D}^2} \simeq \sum_{k=-1}^{\infty} t^{\frac{k-4}{2}} a_k \left(Q, \not{D}^2\right)$$

There is a trick which allows to compute $a_k(Q, \not{D}^2)$ using the known expressions for $a_{k+2}(\tilde{Q}, \mathcal{L})$, where $\mathcal{L}$ is a generic Laplace-type operator and $\tilde{Q}$ is a matrix valued function, see JHEP 1803 (2018) 072 by M.K. and D.Vassilevich.

In our case a_{-1}, a_0, a_1 and a_2 vanish, therefore

$$\begin{aligned}\delta W_{\rm odd} &= -\mathrm{i}\sqrt{\pi} a_3(\delta\not{D}, \not{D}^2) = \int_{\partial\mathcal{M}} d^3x \sqrt{h}\, \varepsilon_\alpha \Big\{ -\frac{\mathrm{i}}{384\pi} (\delta g_{jq}) \tilde{R}_{sp}{}^{qk}{}_{:k} \epsilon^{njsp} \\ &+ \frac{\mathrm{i}}{256\pi} \Big((\delta g_{si})_{;n} K^i_{p:l} - (\delta g_{si}) \left(K^i_l K^r_{p:r} + K^r_p K^i_{l:r} + K^{ri} K_{rp:l} \right) \Big) \epsilon^{nspl}\end{aligned}$$

Gravitational contribution: the 4D the answer.

The solution of the variational equation reads:

$$W^{\text{odd}} = -\frac{\mathrm{i}}{4\pi}\frac{1}{96}\int_{\partial\mathcal{M}} d^3x\sqrt{h}\varepsilon_\alpha\left[\left(\tilde{\Gamma}^r_{qi}\partial_j\tilde{\Gamma}^q_{rk} + \frac{2}{3}\tilde{\Gamma}^r_{qi}\tilde{\Gamma}^q_{pj}\tilde{\Gamma}^p_{rk}\right)\epsilon^{nijk} + \frac{3}{2}K_{si}K^i_{p:l}\,\epsilon^{nspl}\right]$$

- This answer is invariant upon the local Weyl transformations:

$$g_{\mu\nu} \longrightarrow e^{2\phi}g_{\mu\nu}$$

- The coefficient $\frac{1}{96}$ in from of the Chern-Simons term is exactly twice smaller than the corresponding coefficient in the 3D case.
- It has no relation to the (bulk) Pontryagin type topological density, regardless of the choice of the sign factors ϵ_α.

$$\begin{aligned} P &= \frac{1}{4}\int_{\mathcal{M}} d^4x\sqrt{g}\,\epsilon^{\mu\nu\alpha\beta}R^\sigma{}_{\tau\mu\nu}R^\tau{}_{\sigma\alpha\beta} \\ &= -\int_{\partial\mathcal{M}} d^3x\sqrt{h}\left[\left(\tilde{\Gamma}^m_{il}\partial_j\Gamma^l_{km} + \frac{2}{3}\tilde{\Gamma}^l_{im}\tilde{\Gamma}^m_{jp}\tilde{\Gamma}^p_{kl}\right)\epsilon^{nijk} - 2K_{il}K^l_{k:j}\epsilon^{nijk}\right] \end{aligned}$$

Summary.

- We considered the massless QED.

- If one traps fermions inside the $4D$ manifold with a boundary, the one loop radiative corrections induce the Chern-Simons term on the boundary.

- This Chern-Simons term comes out from the spectral asymmetry of the Dirac operator due to the boundary conditions. Presence of such an asymmetry represents the parity anomaly.

- The level of this induced Chern-Simons term is exactly twice smaller than in the 3D case.

- Apart from that the P-odd radiative corrections induce the gravitational Chern-Simons term. The overall coefficient is again twice smaller than in the 3D case. The main novelty in the 4D setup is a presence of the very specific contribution, which depends on the extrinsic curvature.