Reductions of the dispersionless DKP hierarchy

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Introduction

- Integrable hierarchies='solvable' systems with infinitely many variables (e.g., $\mathbf{t} = \{t_1, t_2, \dots\}$)
- Dispersionless integrable hierarchies =quasi-classical limits of certain integrable hierarchies.
- N-variable reduction: solution depend on ∞-many variables t only through N functions,

DKP

- The DKP hierarchy is one of the integrable hierarchies introduced by M.Jimbo and T.Miwa in 1983. It was subsequently rediscovered and came to be also known as
- the coupled KP hierarchy [R.Hirota,Y.Ohta (1991)]
- the Pfaff lattice [M.Adler and others (1999-2002), S.Kakei((1999)]
- Bearing certain similarities with the KP and Toda chain hierarchies, the DKP one is essentially different and less well understood.

Algorithm. One-variable reduction

One-variable reduction: solution depend on ∞-many variables t only through 1 function.

- Start with Hirota equations of the dispersionless hierarchy.
- Introduce some functions to rewrite the equations in a more compact form.
- Easy calculations (take log, ∂_{t_1})
- Consider one-variable reductions of the dispersionless hierarchy.
- The consistency condition for one-variable reductions, Loewner equation
    - dispersionless KP $\Leftrightarrow$ chordal Loewner equation
    - dispersionless Toda $\Leftrightarrow$ radial (i.e. original) Loewner equation
    - dispersionless DKP $\Leftrightarrow$?

The answer:
dispersionless DKP $\Leftrightarrow$ annulus Loewner (Goluzin-Komatu) equation

The dispersionless version of the DKP hierarchy (the dDKP hierarchy) was suggested by Takasaki (2009). It is an infinite system of differential equations for a real-valued function $F = F(\mathbf{t})$ of the infinite number of (real) "times" $\mathbf{t} = \{t_0, t_1, t_2, \ldots\}$.

$$e^{D(z)D(\zeta)F}\left(1 - \frac{1}{z^2\zeta^2}\, e^{2\partial_{t_0}(2\partial_{t_0}+D(z)+D(\zeta))F}\right) = 1 - \frac{\partial_{t_1}D(z)F - \partial_{t_1}D(\zeta)F}{z-\zeta} \tag{1}$$

$$e^{-D(z)D(\zeta)F}\,\frac{z^2 e^{-2\partial_{t_0}D(z)F} - \zeta^2 e^{-2\partial_{t_0}D(\zeta)F}}{z-\zeta} = z + \zeta - \partial_{t_1}\Big(2\partial_{t_0} + D(z) + D(\zeta)\Big)F, \tag{2}$$

where

$$D(z) = \sum_{k\geq 1} \frac{z^{-k}}{k}\,\partial_{t_k}\,.$$

- The differential equations are obtained by expanding these equation in powers of z, ζ.
  For example, the first two equations of the hierarchy are

$$\begin{cases} 6F_{11}^2 + 3F_{22} - 4F_{13} = 12e^{4F_{00}} \\ \\ 2F_{03} + 4F_{01}^3 + 6F_{01}F_{11} - 6F_{01}F_{02} = 3F_{12}. \end{cases} \tag{3}$$

  Here and below we use the short-hand notation
  $F_{mn} \equiv \partial_{t_m}\partial_{t_n}F$.
- The dispersionless KP equation written in the Hirota form

$$6F_{11}^2 + 3F_{22} - 4F_{13} = 0$$

Elliptic parametrization

- It was shown (V.A., A.Zabrodin) that equations (1), (2), when rewritten in an elliptic parametrization in terms of Jacobi's theta-functions $\theta_a(u,\tau)$, assume a nice and suggestive form:

$$(z^{-1}-\zeta^{-1})e^{(\partial_{t_0}+D(z))(\partial_{t_0}+D(\zeta))F}=\frac{\theta_1(u(z)-u(\zeta),\tau)}{\theta_4(u(z)-u(\zeta),\tau)}\,. \quad (4)$$

  Here the function $u(z)$ is defined by

$$e^{\partial_{t_0}(\partial_{t_0}+D(z))F}=z\,\frac{\theta_1(u(z),\tau)}{\theta_4(u(z),\tau)}\,. \quad (5)$$

- The modular parameter τ is a dynamical variable: $\tau=\tau(\mathbf{t})$.
- In what follows we will use the differential operator

$$\nabla(z)=\partial_{t_0}+D(z). \quad (6)$$

- Take the log of (4)

$$\log\left(z_1^{-1} - z_2^{-1}\right) + \nabla(z_1)\nabla(z_2)F = \log\frac{\theta_1(u(z_1)-u(z_2))}{\theta_4(u(z_1)-u(z_2))}. \quad (7)$$

- Introduce the function

$$S(u,\tau) := \log\frac{\theta_1(u,\tau)}{\theta_4(u,\tau)}, \quad (8)$$

- Let's take ∂_{t_0}

$$\partial_{t_0}\nabla(z_1)\nabla(z_2)F = \partial_{t_0}\log\frac{\theta_1(u(z_1)-u(z_2))}{\theta_4(u(z_1)-u(z_2))}. \quad (9)$$

- We get the equation

$$\nabla(z_1)S(u(z_2)|\tau) = \partial_{t_0}S(u(z_1)-u(z_2)|\tau). \quad (10)$$

- We are looking for solutions of the hierarchy such that $u(z, \mathbf{t})$ and $\tau(\mathbf{t})$ depend on the times through a single variable $\lambda = \lambda(\mathbf{t})$.
- Our goal is to characterize the class of functions $u(z, \lambda)$, $\tau(\lambda)$ that are consistent with the structure of the hierarchy and can be used for one-variable reductions.
- It was shown that such one-variable reductions are classified by solutions of a differential equation which is an elliptic analogue of the famous Löwner equation– Goluzin-Komatu equation:

$$4\pi \mathrm{i}\, \partial_\lambda u(z,\lambda) = \Big[-\zeta_1\Big(u(z,\lambda)+\xi(\lambda),\, \tfrac{\tau}{2}\Big) + \zeta_1\Big(\xi(\lambda),\, \tfrac{\tau}{2}\Big)\Big] \frac{\partial \tau}{\partial \lambda}, \tag{11}$$

- We use notation $\zeta_a(u,\tau) := \partial_u \log \theta_a(u,\tau)$
- $\xi(\lambda)$ is an arbitrary (continuous) function of λ (the "driving function").

The formulas simplify a bit if we choose $\lambda = \tau$.

The system of reduced equations and their solution.

In order to complete the description of one-variable reductions, we should derive the equation satisfied by $\tau(\mathbf{t})$ and find its solution.

- Here we use the expansion

$$S(u(z)+\xi) = S(\xi) + \sum_{k\geq 1} \frac{z^{-k}}{k} B_k(\xi),$$

which defines the functions $B_k(u) = B_k(u|\tau)$ and

$$\frac{S'(u(z)+\xi)}{S'(\xi)} = 1 + \sum_{k\geq 1} \frac{z^{-k}}{k} \phi_k(\xi(\tau)|\tau) \tag{12}$$

- In terms of these functions, the equations of the reduced hierarchy are as follows:

$$\frac{\partial \tau}{\partial t_k} = \phi_k(\xi(\tau)|\tau) \frac{\partial \tau}{\partial t_0}, \qquad \phi_k(\xi(\tau)|\tau) := \frac{B_k'(\xi(\tau)|\tau)}{S'(\xi(\tau)|\tau)}, \quad k \geq 1. \tag{13}$$

The system of reduced equations and their solution.

- The common solution to these equations can be written in the hodograph form:

$$\sum_{k=1}^{\infty} t_k \phi_k(\xi(\tau)|\tau) = \Phi(\tau), \tag{14}$$

where $\Phi(\tau)$ is an arbitrary function of τ.

One-variable reduction, summary

- Start with equation of the dispersionless DKP hierarchy.

$$\nabla(z_1)S(u(z_2)|\tau) = \partial_{t_0} S(u(z_1) - u(z_2)|\tau).$$

- Consider one-variable reductions: $\tau(\mathbf{t})$, $u(z, \mathbf{t}) = u(z, \tau(\mathbf{t}))$.
- Find the consistency condition for one-variable reductions (annulus Loewner (Goluzin-Komatu) equation).

$$4\pi \mathrm{i}\, \partial_\tau u(z, \lambda) = -\, \zeta_1\Big(u(z, \lambda) + \xi(\lambda),\, \tfrac{\tau}{2}\Big) + \zeta_1\Big(\xi(\lambda),\, \tfrac{\tau}{2}\Big), \tag{15}$$

- Find the common solution of the system of reduced equations

$$\frac{\partial \tau}{\partial t_k} = \phi_k(\xi(\tau)|\tau)\, \frac{\partial \tau}{\partial t_0}$$

in hodograph form

$$\sum_{k=1}^{\infty} t_k \phi_k(\xi(\tau)|\tau) = \Phi(\tau), \tag{16}$$

- We study diagonal N-variable reductions of the dDKP hierarchy when u depends on the times through N real variables λ_j.
- The starting point is the system of N elliptic Löwner equations which characterize the dependence of $u(z)$ on the variables λ_j: $\{\lambda_i\} = \{\lambda_1, \ldots, \lambda_N\}$

$$4\pi\mathrm{i}\,\partial_{\lambda_j} u(z, \{\lambda_i\}) = \left[-\zeta_1\left(u+\xi_j\,,\, \tfrac{\tau}{2}\right) + \zeta_1\left(\xi_j\,,\, \tfrac{\tau}{2}\right)\right] \frac{\partial \tau}{\partial \lambda_j}, \tag{17}$$

- ξ_j and τ are functions of $\{\lambda_i\}$: $\xi_j = \xi_j(\{\lambda_i\})$
- $\tau = \tau(\{\lambda_i\})$.
- We assume that ξ_j are real-valued functions.
- Their compatibility condition is expressed as the elliptic Gibbons-Tsarev system

The Gibbons-Tsarev system is the compatibility condition for the system of elliptic Löwner equations:

$$\frac{\partial u}{\partial \lambda_j} = \frac{1}{4\pi \mathrm{i}}\left(-\zeta_1\left(u+\xi_j, \frac{\tau}{2}\right) + \zeta_1\left(\xi_j, \frac{\tau}{2}\right)\right)\frac{\partial \tau}{\partial \lambda_j}. \tag{18}$$

- The compatibility condition is

$$F_{jk}(u) := \frac{\partial}{\partial \lambda_j}\frac{\partial u}{\partial \lambda_k} - \frac{\partial}{\partial \lambda_k}\frac{\partial u}{\partial \lambda_j} = 0.$$

- The left hand side is of the form

$$F_{jk}(u) = F^{(1)}_{jk}\frac{\partial \xi_k}{\partial \lambda_j}\frac{\partial \tau}{\partial \lambda_k} - F^{(1)}_{kj}\frac{\partial \xi_j}{\partial \lambda_k}\frac{\partial \tau}{\partial \lambda_j} + F^{(2)}_{jk}\frac{\partial^2 \tau}{\partial \lambda_j \partial \lambda_k} + G_{jk}\frac{\partial \tau}{\partial \lambda_j}\frac{\partial \tau}{\partial \lambda_k}. \tag{19}$$

The coefficients are:

$$F_{jk}^{(1)} = \frac{1}{4\pi\mathrm{i}}\Big(\wp_1(u+\xi_k),\tau') - \wp_1(\xi_k,\tau')\Big),$$

$$F_{jk}^{(2)} = \frac{1}{4\pi\mathrm{i}}\Big(-\zeta_1(u+\xi_k,\tau')+\zeta_1(\xi_k,\tau')+\zeta_1(u+\xi_j,\tau')-\zeta_1(\xi_j,\tau')\Big),$$

$$G_{jk} = \frac{1}{2(4\pi\mathrm{i})^2}\Big(\wp_1'(u+\xi_k,\tau')-\wp_1'(u+\xi_j,\tau')-\wp_1'(\xi_k,\tau')+\wp_1'(\xi_j,\tau')\Big)$$

$$+\ \frac{1}{(4\pi\mathrm{i})^2}\Big(\zeta_1(u+\xi_k,\tau') - \zeta_1(u+\xi_j,\tau') + \zeta_1(\xi_j,\tau')\Big)\wp_1(u+\xi_k,\tau')$$

$$-\ \frac{1}{(4\pi\mathrm{i})^2}\Big(\zeta_1(u+\xi_j,\tau') - \zeta_1(u+\xi_k,\tau') + \zeta_1(\xi_k,\tau')\Big)\wp_1(u+\xi_j,\tau')$$

$$+\ \frac{1}{(4\pi\mathrm{i})^2}\Big(-\zeta_1(\xi_k,\tau')\wp_1(\xi_k,\tau') + \zeta_1(\xi_j,\tau')\wp_1(\xi_j,\tau')\Big),$$

where

- $\wp_a(x,\tau) = -\partial_x\zeta_a(x,\tau)$,
- $\wp_a'(x,\tau) = \partial_x\wp_a(x,\tau)$.

- We get the elliptic analogue of the famous Gibbons-Tsarev system:

$$\frac{\partial \xi_k}{\partial \lambda_j} = \frac{1}{4\pi \mathrm{i}} \left(\zeta_1(-\xi_k + \xi_j, \tau') - \zeta_1(\xi_j, \tau') \right) \frac{\partial \tau}{\partial \lambda_j}, \tag{20}$$

$$\frac{\partial^2 \tau}{\partial \lambda_k \partial \lambda_j} = \frac{1}{2\pi \mathrm{i}} \wp_1(\xi_k - \xi_j, \tau') \frac{\partial \tau}{\partial \lambda_k} \frac{\partial \tau}{\partial \lambda_j} \tag{21}$$

for all $j = 1, \ldots, N$, $j \neq k$.

- The dependence of the λ_j's on $\mathbf{t}$ is given by the equation

$$\nabla(z)\lambda_j = \frac{S'(u(z)+\xi_j)}{S'(\xi_j)}\,\frac{\partial \lambda_j}{\partial t_0}. \qquad (22)$$

- This equation contains an infinite system of partial differential equations of hydrodynamic type.
- We introduce elliptic Faber functions $\Phi_k(u)$ via the expansion $S(u(z)+w) = S(w) + \sum_{k=1}^{\infty} \frac{z^{-k}}{k}\,\Phi_k(w)$ or

$$S'(u(z)+w) = S'(w) + \sum_{k=1}^{\infty} \frac{z^{-k}}{k}\,\Phi_k'(w) \qquad (23)$$

(here $\Phi_k'(w) = \partial_w \Phi_k(w)$). Then the system (22) reads

$$\frac{\partial \lambda_j}{\partial t_k} = \phi_{j,k}(\{\lambda_i\})\,\frac{\partial \lambda_j}{\partial t_0}\,, \qquad \phi_{j,k} = \frac{\Phi_k'(\xi_j)}{S'(\xi_j)}, \qquad (24)$$

Generalized hodograph method

- We have reduced the dDKP hierarchy to the system of elliptic Löwner equations and the auxiliary equations

$$\frac{\partial \lambda_i(\mathbf{t})}{\partial t_n} = \phi_{i,n}(\{\lambda_j(\mathbf{t})) \frac{\partial \lambda_i(\mathbf{t})}{\partial t_0}, \qquad (25)$$

  where $\phi_{i,n}$ are as in (24).
- Step 1 to show that this system of equations is consistent.
- Step 2 to show that it can be solved by Tsarev's generalized hodograph method.

Generalized hodograph method

- As is easy to see, the compatibility condition of the system (25) is

$$\frac{\partial_{\lambda_j}\phi_{i,n}}{\phi_{j,n}-\phi_{i,n}} = \frac{\partial_{\lambda_j}\phi_{i,n'}}{\phi_{j,n'}-\phi_{i,n'}} \qquad \text{for all } i \neq j,\ n, n'.$$

- We should show that

$$\Gamma_{ij} := \frac{\partial_{\lambda_j}\phi_{i,n}}{\phi_{j,n}-\phi_{i,n}} \tag{26}$$

does not depend on n.

Generalized hodograph method

- Consider the following system for $R_i = R_i(\{\lambda_j\})$, $i = 1, \dots, N$:

$$\frac{\partial R_i}{\partial \lambda_j} = \Gamma_{ij}(R_j - R_i), \qquad i, j = 1, \dots, N, \quad i \neq j, \tag{27}$$

where Γ_{ij} is defined as

$$\Gamma_{ij} = -\frac{1}{4\pi \mathrm{i}} \frac{S'(\xi_j)}{S'(\xi_i)} S''(\xi_i - \xi_j) \frac{\partial \tau}{\partial \lambda_j}. \tag{28}$$

(when $N = 1$, the condition (27) is void).

- Then the following holds:
  - (i) The system (27) is compatible.
  - (ii) Assume that R_i satisfy the system (27). If $\lambda_i(\mathbf{t})$ is defined implicitly by the hodograph relation

$$t_0 + \sum_{n \geq 1} \phi_{i,n}(\{\lambda_j\}) t_n = R_i(\{\lambda_j\}), \tag{29}$$

then $\lambda_j(\mathbf{t})$ satisfy (25).

- In fact the compatibility conditions of the system (27) are

$$\frac{\partial \Gamma_{ij}}{\partial \lambda_k} = \frac{\partial \Gamma_{ik}}{\partial \lambda_j}, \qquad i \neq j \neq k, \tag{30}$$

(which is the Tsarev compatibility condition), together with

$$\frac{\partial \Gamma_{ij}}{\partial \lambda_k} = \Gamma_{ij}\Gamma_{jk} + \Gamma_{ik}\Gamma_{kj} - \Gamma_{ik}\Gamma_{ij}, \qquad i \neq j \neq k. \tag{31}$$

N-variable reductions, summary

- We start with the system of N elliptic Löwner equations which characterize the dependence of $u(z)$ on the variables λ_j:

$$4\pi\mathrm{i}\,\partial_{\lambda_j} u(z,\{\lambda_i\}) = \left[-\,\zeta_1\left(u+\xi_j\,,\,\tfrac{\tau}{2}\right) + \zeta_1\left(\xi_j\,,\,\tfrac{\tau}{2}\right)\right]\frac{\partial\tau}{\partial\lambda_j}, \tag{32}$$

- Their compatibility condition is expressed as the elliptic Gibbons-Tsarev system

$$\frac{\partial\xi_k}{\partial\lambda_j} = \frac{1}{4\pi\mathrm{i}}\left(\zeta_1(-\xi_k+\xi_j,\tau') - \zeta_1(\xi_j,\tau')\right)\frac{\partial\tau}{\partial\lambda_j}, \tag{33}$$

$$\frac{\partial^2\tau}{\partial\lambda_k\partial\lambda_j} = \frac{1}{2\pi\mathrm{i}}\,\wp_1(\xi_k-\xi_j,\tau')\frac{\partial\tau}{\partial\lambda_k}\frac{\partial\tau}{\partial\lambda_j} \tag{34}$$

for all $j = 1,\ldots,N$, $j \neq k$.

N-variable reductions, summary

- We have reduced the dDKP hierarchy to the system of elliptic Löwner equations and the auxiliary equations

$$\frac{\partial \lambda_i(\mathbf{t})}{\partial t_n} = \phi_{i,n}(\{\lambda_j(\mathbf{t})\}) \frac{\partial \lambda_i(\mathbf{t})}{\partial t_0}, \tag{35}$$

  where $\phi_{i,n}$ are as in (24).

- We show that

$$\Gamma_{ij} := \frac{\partial_{\lambda_j} \phi_{i,n}}{\phi_{j,n} - \phi_{i,n}} \tag{36}$$

  does not depend on n and this system of equations is consistent.

- We show that it can be solved by Tsarev's generalized hodograph method.

N-variable reductions, summary

For this we

- Consider the following system for $R_i = R_i(\{\lambda_j\})$, $i = 1, \ldots, N$:

$$\frac{\partial R_i}{\partial \lambda_j} = \Gamma_{ij}(R_j - R_i), \qquad i, j = 1, \ldots, N, \quad i \neq j, \tag{37}$$

  where Γ_{ij} is defined as

$$\Gamma_{ij} = -\frac{1}{4\pi \mathrm{i}} \frac{S'(\xi_j)}{S'(\xi_i)} S''(\xi_i - \xi_j) \frac{\partial \tau}{\partial \lambda_j}. \tag{38}$$

- Then we chek that :
  (i) The system (37) is compatible:

$$\frac{\partial \Gamma_{ij}}{\partial \lambda_k} = \frac{\partial \Gamma_{ik}}{\partial \lambda_j}, \qquad i \neq j \neq k, \tag{39}$$

  together with

$$\frac{\partial \Gamma_{ij}}{\partial \lambda_k} = \Gamma_{ij}\Gamma_{jk} + \Gamma_{ik}\Gamma_{kj} - \Gamma_{ik}\Gamma_{ij}, \qquad i \neq j \neq k. \tag{40}$$

(ii) Assume that R_i satisfy the system (37). If $\lambda_i(\mathbf{t})$ is defined implicitly by the hodograph relation

$$t_0 + \sum_{n\geq 1} \phi_{i,n}(\{\lambda_j\}) t_n = R_i(\{\lambda_j\}), \tag{41}$$

then $\lambda_j(\mathbf{t})$ satisfy

$$\frac{\partial \lambda_i(\mathbf{t})}{\partial t_n} = \phi_{i,n}(\{\lambda_j(\mathbf{t})) \frac{\partial \lambda_i(\mathbf{t})}{\partial t_0}, \tag{42}$$

.

Thank you for attention!

- In what follows we use the differential operator

$$\nabla(z) = \partial_{t_0} + D(z). \tag{43}$$

- Introducing the functions

$$p(z) = z - \partial_{t_1}\nabla(z)F, \qquad w(z) = z^2 e^{-2\partial_{t_0}\nabla(z)F}, \tag{44}$$

- we can rewrite equations (1), (2) in a more compact form

$$e^{D(z)D(\zeta)F}\left(1 - \frac{1}{w(z)w(\zeta)}\right) = \frac{p(z) - p(\zeta)}{z - \zeta}, \tag{45}$$

$$e^{-D(z)D(\zeta)F + 2\partial_{t_0}^2 F}\,\frac{w(z) - w(\zeta)}{z - \zeta} = p(z) + p(\zeta). \tag{46}$$

Multiplying the two equations, we get the relation

$$p^2(z) - e^{2F_{00}}\Big(w(z) + w^{-1}(z)\Big) = p^2(\zeta) - e^{2F_{00}}\Big(w(\zeta) + w^{-1}(\zeta)\Big)$$

from which it follows that $p^2(z) - e^{2F_{00}}\Big(w(z) + w^{-1}(z)\Big)$ does not depend on z (here and below we use the short-hand notation $F_{mn} = \frac{\partial^2 F}{\partial t_m \partial t_n}$). Tending z to infinity, we find that this expression is equal to $F_{02} - 2F_{11} - F_{01}^2$. Therefore, we conclude that $p(z), w(z)$ satisfy the algebraic equation

$$p^2(z) = R^2\Big(w(z) + w^{-1}(z)\Big) + V\,, \tag{47}$$

where

$$R = e^{F_{00}}, \qquad V = F_{02} - 2F_{11} - F_{01}^2. \tag{48}$$

are real numbers depending on the times (R is positive).

- This equation defines an elliptic curve, with w, p being algebraic functions on this curve.

- A natural further step is to uniformize the curve through elliptic functions. We use the standard Jacobi theta functions $\theta_a(u) = \theta_a(u, \tau)$ $(a = 1, 2, 3, 4)$. The elliptic parametrization of (47) is as follows:

$$w(z) = \frac{\theta_4^2(u(z))}{\theta_1^2(u(z))}, \qquad p(z) = \gamma\, \theta_4^2(0)\, \frac{\theta_2(u(z))\, \theta_3(u(z))}{\theta_1(u(z))\, \theta_4(u(z))}, \tag{49}$$

  where $u(z) = u(z, \mathbf{t})$ is some function of z, γ is a z-independent factor.

- $$R = \gamma\, \theta_2(0)\, \theta_3(0), \qquad V = -\gamma^2 \Big(\theta_2^4(0) + \theta_3^4(0)\Big). \tag{50}$$

  Here γ is an arbitrary real parameter but we will see that it is a dynamical variable, as well as the modular parameter τ: $\gamma = \gamma(\mathbf{t})$, $\tau = \tau(\mathbf{t})$.

- It is convenient to normalize $u(z)$ by the condition $u(\infty) = 0$, with the expansion around ∞ being

$$u(z, \mathbf{t}) = \frac{c_1(\mathbf{t})}{z} + \frac{c_2(\mathbf{t})}{z^2} + \dots \tag{51}$$

with real coefficients c_i.

- This identity allows us to represent the equations

$$e^{D(z)D(\zeta)F}\left(1-\frac{1}{w(z)w(\zeta)}\right)=\frac{p(z)-p(\zeta)}{z-\zeta}\,, \qquad (52)$$

$$e^{-D(z)D(\zeta)F+2\partial_{t_0}^2 F}\ \frac{w(z)-w(\zeta)}{z-\zeta}=p(z)+p(\zeta). \qquad (53)$$

as a single equation:

$$\left(z_1^{-1}-z_2^{-1}\right)e^{\nabla(z_1)\nabla(z_2)F}=\frac{\theta_1(u(z_1)-u(z_2))}{\theta_4(u(z_1)-u(z_2))}\,. \qquad (54)$$