Non-vanishing of vacuum diagrams in light-cone perturbation theory

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ABSTRACT: Recently, J. Collis has pointed out that vacuum diagrams are, contrary to the general belief, non-vanishing in light-front field theory. In our contribution, we first recall the old (forgotten) arguments by Chang and Ma and by Yan, why this should be so. Then we apply the argument of analyticity of the self-energy diagrams in $\lambda\phi^3$ and $\lambda\phi^4$ two-dimensional models in light-front (LF) perturbation theory to calculate the vacuum bubbles explicitly as $p = 0$ values of the appropriate self-energy diagrams. The results are non-zero and agree with the usual Feynman-diagram calculation. Surprisingly, the light-front bubbles are non-vanishing NOT due to LF zero modes. This is confirmed by the DLCQ calculations, where the mode with $n = 0$ (the LF zero mode with $k^+ = 0$) is manifestly absent, but the results still converge to the continuum values for increasing "harmonic resolution" K. Generalization to realistic 3+1 dimensional case and to e.g. Yukawa theory is straightforward.

I. INTRODUCTION

Dirac RMP 1949:

three forms of the Hamiltonian relativistic dynamics

front form the most efficient one, only 3 dynamical Poincaré generators, physical vacuum obtained kinematically (no need to solve the dynamics as in the conventional ("instant" or space-like (SL) form), positivity of the (kinematical) quantity - the LF momentum p^+ in addition to the (LF) energy P^-.

Here $p^{\pm} = p^0 \pm p^3, p^- = \frac{p_\perp^2 + m^2}{p^+}$ - no square-root ambiguity, $\partial_\mu \partial^\mu = \partial_+ \partial_- - \partial_\perp^2 \Rightarrow$ different structure of field equation, smaller number of dynamical dofs (constraints), etc.

Quantum field theory formulated in terms of light-front (LF) variables:

few unusual features: indications of inconsistency?

Example: massless fields in 2 dimensions: seemingly hard to initialize, quantization appeared obscure, ad hoc constructions...

In fact they emerge as the massless limit of massive fields (scalar, fermion...). Based on the massive 2-point functions, change of variables for some components. A consistent scheme, correct consequences (solvable models, bosonization, conformal field theory...)

Another problem appeared to be paradoxically related to the most celebrated property of the LF quantization - vacuum simplicity

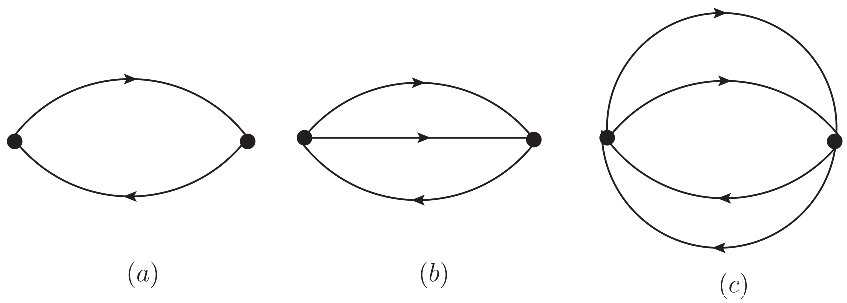
Well known: positivity of the LF momentum p^+ together with its conservation implies that the ground state of any dynamical model cannot contain quanta carrying $p^+ \neq 0$. Only a tiny subset of all field modes, namely those carrying $p^+ = 0$ - the LF zero modes (ZM) - can contribute

NB: some field modes (ZM of the scalar field) which appear as dynamical ones in the conventional ("space-like", SL for short) theory become constrained (non-dynamical) in the LF form of the theory $\Rightarrow$ cannot contribute to vacuum processes directly

QUESTIONS: how does LF theory describe vacuum phenomena? Is the LF dynamics equivalent to the SL one? Can it predict something new?

Prevailing opinion (Brodsky, Burkhard...): LF vacuum always "trivial" (empty state), in particular: vacuum bubbles do not exist in LF perturbation theory, cosmological consequences (Brodsky and Schrock) if true would mean LF theory is not equivalent to the SL theory

Recently, **J. Collins** has pointed out this controversy along with a corrected treatment for the simplest LF vacuum loop with 2 internal lines, identified a mathematical difficuty



Vacuum bubbles for ϕ^3 and ϕ^4 models

The equivalence issue realized and studied already in the pioneering papers on LF perturbative S-matrix by Cheng and Ma (1969) and by T.-M. Yan (1973) includig the vacuum problem at the perturbation theory level

Method: covariant Feynman amplitudes (integrals) rewritten in terms of LF variables, **the delicate step:** to perform the integration in p^- variable, since the propagators in 2D behave as $(k^+k^- - m^2 + i\epsilon)^{-1}$ instead of $(k_0^2 - k_1^2 - m^2 + i\epsilon)^{-1}$ - convergence

T.-M. Yan, PRD 7, 1780 (1973): $I = \int d^4p \frac{1}{(p^2-\mu^2+i\epsilon)^3} = \frac{\pi^2}{2i\mu^2}$.
Here $d^4p = dp^0dp^1dp^2dp^3$ and $p^0 \to idp^4$. In LF variables,

$$I = \int dp^+dp^-d^2p_\perp \frac{1}{(p^+p^- - p_\perp^2 - \mu^2 + i\epsilon)^3} = -\frac{\pi}{4}\int dp^+dp^- \frac{1}{(p^+p^- - \mu^2 + i\epsilon)^2}. \qquad (1)$$

A double pole at $p^- = \frac{\mu^2 - i\epsilon}{p^+}$, at infinity for $p^+ = 0$, a careful treatment needed:

$$I = -\frac{\pi}{4}\int\limits_{-\infty}^{+\infty} dp^+ \lim_{\Lambda\to\infty}\int\limits_{-\Lambda}^{+\Lambda} dp^- \frac{1}{(p^+p^- - \mu^2 + i\epsilon)^2} =$$

$$= \frac{\pi}{4}\int\limits_{-\infty}^{+\infty} \frac{dp^+}{p^+} \lim_{\Lambda\to\infty}\Big(\frac{1}{p^+\Lambda - \mu^2 + i\epsilon} - \frac{1}{-p^+\Lambda - \mu^2 + i\epsilon}\Big). \qquad (2)$$

Using the identity

$$\frac{1}{p^+}\Big(\frac{1}{p^+\Lambda-\mu^2+i\epsilon}-\frac{1}{-p^+\Lambda-\mu^2+i\epsilon}\Big)=\frac{1}{\mu^2}\Big(\frac{\Lambda}{p^+\Lambda-\mu^2+i\epsilon}-\frac{\Lambda}{p^+\Lambda+\mu^2-i\epsilon}\Big), \tag{3}$$

for $\Lambda\to\infty$, one gets

$$I=\frac{\pi}{4\mu^2}\int\limits_{-\infty}^{+\infty}dp^+\Big(\frac{1}{p^++i\epsilon}-\frac{1}{p^+-i\epsilon}\Big)=\frac{\pi}{4\mu^2}\int\limits_{-\infty}^{+\infty}dp^+\big[-2i\pi\delta(p^+)\big]=\frac{\pi^2}{2i\mu^2}. \tag{4}$$

Same result with the exponential α-representation $iD^{-1}=\int\limits_0^\infty d\alpha e^{i\alpha(D+i\epsilon)}$.

Chang and Ma different method for a vacuum bubble with 3 internal lines

$$V=\int dp^+dp^-\frac{1}{p^+p^--\mu^2+i\epsilon}\Sigma(p^+p^-), \tag{5}$$

where $\Sigma(p^2)$ represented as

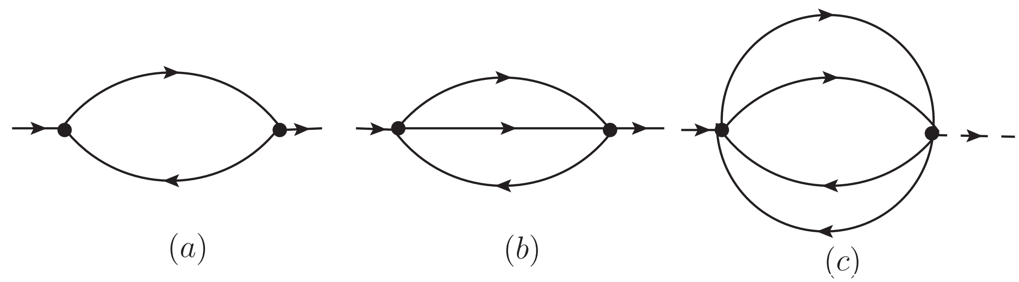
$$\Sigma(p^2) = \int d\lambda F(\lambda) e^{i\lambda p^2}, \;\; F(\lambda) = \int\limits_0^{+\infty} d\alpha_1 d\alpha_2 \delta(\lambda(\alpha_1 + \alpha_2) - \alpha_1\alpha_2) e^{-i\mu^2\alpha_1\alpha_2/\lambda}, \tag{6}$$

where the above α-representation used here and also in (5). Insert Σ of (6) into (5):

$$V = \int dp^+ dp^- \big[-i \int\limits_0^{+\infty} d\alpha d\lambda F(\lambda) e^{ip^2(\lambda+\alpha) - i\mu^2\alpha} \big] =$$

$$= \int dp^+ \Big[-2\pi i \int_0^{+\infty} d\alpha d\lambda F(\lambda)(\lambda + \alpha)^{-1} e^{-i\mu^2\alpha} \Big] \delta(p^+). \tag{7}$$

Non-zero result, but no explicit formula given.



Self-energy diagrams for for ϕ^3, ϕ^4 and ϕ^5 models

HERE: generalization of Collin's analysisi to loops with more internal lines, using the analyticity argument, both continuum and finite-volume formulation (DLCQ), complete agreement with covariant Feynman results

II. THE FORMALISM AND SIMPLE EXAMPLES

The basic formula for the S-matrix in the "old-fashioned", Hamiltonian, LF-time ordered, non-manifestly covariant PT (it avoids the k^- integration in a natural way, also: energy denominators instead of covariant propagators)

With $V = P^-_{int}$, $\quad V(x^+) = e^{\frac{i}{2}P^-_0 x^+} V(0) e^{-\frac{i}{2}P^-_0 x^+}$, we have

$$S_{fi} = \delta_{fi} - \frac{i}{2} \int\limits_{-\infty}^{+\infty} dx^+ \langle \phi_f | V(x^+) | \phi_i \rangle -$$

$$-\frac{1}{4} \int\limits_{-\infty}^{+\infty} dx_1^+ \langle \phi_f | V(x_1^+) | \phi_n \rangle \int\limits_{-\infty}^{x_1^+} dx_2^+ \langle \phi_n | V(x_2^+) | \phi_i \rangle + \ldots, \tag{8}$$

The T and M matrices are defined after extracting kinematical factors:

$$S_{fi} = \delta_{fi} - 2\pi i \delta(p_i^- - p_f^-) T_{fi}, \quad T_{fi} = \frac{1}{\sqrt{p_f^+ p_i^+}} \delta(p_f^+ - p_i^+) M_{fi} \tag{9}$$

A complete set of states was inserted in (8):

$$\hat{1} = \sum_n |\phi_n\rangle\langle\phi_n| = |0\rangle\langle 0| + \int_0^{+\infty} dl_1^+ a^\dagger(l_1^+)|0\rangle\langle 0|a(l_1^+) +$$

$$+ \int_0^{+\infty} dl_1^+ \int_0^{+\infty} dl_2^+ a^\dagger(l_2^+) a^\dagger(l_1^+)|0\rangle\langle 0|a(l_1^+)a(l_2^+) + \dots . \quad (10)$$

We shall work with $\lambda\phi^3$ and $\lambda\phi^4$ models in 2D, for which

$$P_{int}^- = \frac{\lambda}{3!} 3 \int_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \int_0^{+\infty} \frac{dp^+}{\sqrt{4\pi p^+}} \int_0^{+\infty} \frac{dq^+}{\sqrt{4\pi q^+}} 2\pi\delta(p^+ + k^+ - q^+)$$

$$\times \left\{ a^\dagger(q^+)a(k^+)a(p^+) + a^\dagger(p^+)a^\dagger(k^+)a(q^+) \right\} \quad (11)$$

$$P^-_{int} = V_1 + V_2 + V_3 =$$

$$= \frac{\lambda}{4!} \int\limits_0^{+\infty} \frac{dk^+}{\sqrt{4\pi k^+}} \int\limits_0^{+\infty} \frac{dp^+}{\sqrt{4\pi p^+}} \int\limits_0^{+\infty} \frac{dq^+}{\sqrt{4\pi q^+}} \int\limits_0^{+\infty} \frac{dr^+}{\sqrt{4\pi r^+}} 8\pi$$

$$\times \Big\{ \Big[a^\dagger(k^+)a^\dagger(p^+)a^\dagger(q^+)a(r^+) + a^\dagger(r^+)a(p^+)a(q^+)a(k^+) \Big] \delta(k^+ + p^+ + q^+ - r^+)$$

$$+ \frac{3}{2} a^\dagger(k^+)a^\dagger(p^+)a(q^+)a(r^+)\delta(k^+ + p^+ - q^+ - r^+) \Big\}. \qquad (12)$$

The rules of the LF perturbation theory imply that the vacuum amplitudes (bubbles) vanish (or rather are mathematically ill-defined) (Yan 1973) as the corresponding integrals contain the delta function $\delta(p^+_1 + p^+_2 ... + p^+_n)$ (momentum conservation) which can be satisfied only if all of them vanish, leading to singular integrands. The simplest example: **LF tadpole**

$$a^\dagger(k^+_1)a(k^+_2)a^\dagger(k^+_3)a(k^+_4) = \delta(k^+_2 - k^+_3)a^\dagger(k^+_1)a(k^+_4) + no \equiv V_T + no, \quad (13)$$

- it arises in the process of normal-ordering the Hamiltonian

$$S_{fi}^{(1)} = -\frac{i}{2}\int\limits_{-\infty}^{+\infty} dx^+ \langle 0|a(p_f^+)e^{\frac{i}{2}p_f^- x^+} V_T e^{-\frac{i}{2}p_i^- x^+} a^\dagger(p_i^+)|0\rangle \qquad (14)$$

$$\Rightarrow M_T = \frac{\lambda}{8\pi}\int\limits_0^{+\infty}\frac{dk^+}{k^+} \rightarrow \frac{\lambda}{8\pi}\int\limits_\epsilon^\Lambda \frac{dk^+}{k^+} = \frac{\lambda}{8\pi}\int\limits_{\frac{\mu^2}{\Lambda}}^{\Lambda}\frac{dk^-}{k^-} = \frac{\lambda}{8\pi}\log\frac{\Lambda^2}{\mu^2}. \qquad (15)$$

change of variable $k^+ \rightarrow \frac{\mu^2}{k^+}$ performed, no need to hunt for poles at infinity

- A. Harindranath, L. Martinovic and J. P. Vary, PRD 64, 105016 (2001):

IMF, near-LC and LFPT loop diagrams (self-energy and scattering): comparison in particular, one-loop self-energy in $\lambda\phi^3(3+1)$ toy model

$$\Sigma(p^2) = \frac{\lambda^2}{4(2\pi)^3}\int_0^1 dx \int d^2q_\perp \frac{1}{p^2x(1-x)-(q_\perp)^2-\mu^2+i\epsilon}. \tag{16}$$

Reducing to 1+1 dim and setting $p=0$ (=vacuum bubble, J. Collin's case), we have

$$V \equiv \Sigma(0) = \frac{\lambda^2}{8\pi}\int_0^1 dx \frac{1}{-\mu^2+i\epsilon} = -\frac{1}{8\pi}\frac{\lambda^2}{\mu^2}. \tag{17}$$

We did not realize this connection at that time.

Simple case - analytic formula for $s \equiv p^2 \neq 0$:

$$V = \frac{\lambda^2}{8\pi} \int\limits_0^1 \frac{dx}{sx(1-x) - \mu^2 + i\epsilon} = -4 \frac{\arctan \sqrt{\frac{s}{4\mu^2 - s}}}{\sqrt{4\mu^2 s(1-s)}} \frac{\lambda^2}{8\pi}. \qquad (18)$$

Undefined for $s = 0$, L'Hospital yields the correct value $\sim -1/\mu^2$.

Vacuum amplitudes in the SL form: bubble in ϕ^3 toy model

The corresponding Feynman rules lead to the double two-dimensional integral expression

$$V_3(\mu) = \int d^2p \int d^2q \frac{1}{(p^2 - \mu^2 + i\epsilon)(q^2 - \mu^2 + i\epsilon)((p+q)^2 - \mu^2 + i\epsilon)}. \qquad (19)$$

Can be evaluated in a few ways: by using the Feynman parameters, by means of α-representation or via more sophisticated mathematical methods (Mellin-Barnes representation for powers of massive propagators (Davydychev and Tausk, NPB (1993), PRD (1996)) - the same result

$$V(\mu) = -\frac{C}{\mu^2}, \quad C = 2.344..., \tag{20}$$

The constant C has a particular representation in each of the methods

The first method: combine the propagators into one denominator by means of the auxiliary integrals in terms of the Feynman parameters x_i, then go over to Euclidean space and calculate the integrals in p and q variables. The result is the the double-integral representation

$$V_3 = \frac{\pi^2}{\mu^2}\int_0^1 dx_1 dx_2 dx_3 \frac{\delta(1-x_1-x_2-x_3)}{x_1x_2+x_1x_3+x_2x_3}, \tag{21}$$

which can be transformed by a suitable change of variables to the LF integral with $p = 0$! (see below)

III. LIGHT-FRONT CALCULATION IN CONTINUUM

The result (20) obtained in a very simple way also in the LFPT, contrary to the the general belief

Naively, the LFPT rules yield (Yan)

$$\tilde{V} \sim \int_0^\infty \frac{dp_1^+}{p_1^+} \int_0^\infty \frac{dp_2^+}{p_2^+} \int_0^\infty \frac{dp_3^+}{p_3^+} \frac{\delta(p_1^+ + p_2^+ + p_3^+)}{(-\mu^2)[\frac{1}{p_1^+} + \frac{1}{p_2^+} + \frac{1}{p_3^+}]}. \tag{22}$$

This essentially expresses the fact that since the incoming momentum is zero, the conservation of the LF momentum p^+ requires that each of three internal lines must also carry vanishing LF momentum.

THE CORRECT METHOD: start with the (self-energy) graph with nonvanishing external momentum and write down the corresponding LF amplitude. The expected analyticity in p then permits one to consider its value at $p = 0$ (after going over to relative LF momenta – covariant form); the vacuum loop emerges simply as the limit of the corresponding self-energy graph for vanishing external momentum:

$$\Sigma(p) = N \int_0^{p^+} \frac{dk^+}{k^+} \int_0^{p^+-k^+} \frac{dl^+}{l^+(p^+ - k^+ - l^+)} \frac{1}{p^- - \frac{\mu^2}{k^+} - \frac{\mu^2}{l^+} - \frac{\mu^2}{p^+-k^+-l^+} + i\epsilon}. \tag{23}$$

Introducing the dimensionless variables $x = \frac{k^+}{p^+}$, $y = \frac{l^+}{p^+}$, $\Sigma(p)$ becomes

$$\Sigma(p) = N \int_0^1 \frac{dx}{x} \int_0^{1-x} \frac{dy}{y(1-x-y)} \frac{1}{\left[p^2 - \frac{\mu^2}{x} - \frac{\mu^2}{y} - \frac{\mu^2}{1-x-y}\right]}. \tag{24}$$

Now we can set $p = 0$. This expression replaces the incorrect Eq.(22). The integral over the variable y can be performed explicitly, yielding

$$\Sigma(0) = -\frac{2}{\mu^2} \int_0^1 dx \frac{\ln \frac{\sqrt{1-x}+\sqrt{1+3x}}{\sqrt{1-x}-\sqrt{1+3x}}}{\sqrt{(1-x)(1+3x)}} \tag{25}$$

The numerical computation

$$\Sigma(0) = -C/\mu^2, \quad C = 2,344\ldots \tag{26}$$

FINITE VOLUME (DLCQ) CALCULATION

Remarkably, the same result obtained in the discretized (finite-volume) treatment with (anti-)periodic boundary conditions (BC). In both cases, the field mode carrying $p^+ = 0$ is manifestly absent.

The corresponding field expansion at $x^+ = 0$ is

$$\phi(0, x^-) = \frac{1}{\sqrt{2L}} \sum_n^{\infty} \frac{1}{\sqrt{p_n^+}} \left[a_n e^{-ip_n^+ x^-} + a_n^\dagger e^{ip_n^+ x^-} \right], \qquad (27)$$

where $p_n^+ = 2\pi n/L$ and L is the lenght of the finite interval. The index n runs over half-integers for antiperiodic boundary conditions and over integers for periodic BC, with $n = 0$ excluded. Reason: this field mode is not a dynamical quantity, but a constrained variable, expressed in terms of the $n \neq 0$ field modes. The DLCQ analog of the $\Sigma(p)$ amplitude is

$$\Sigma(p) = -\lambda^2 N \sum_{q^+}^{p^+} \sum_{k^+}^{p^+-q^+} \frac{1}{k^+ q^+ (p^+ - k^+ - q^+) \left[p^- - \frac{\mu^2}{k^+} - \frac{\mu^2}{q^+} - \frac{\mu^2}{p^+-k^+-q^+}\right]}. \tag{28}$$

For $p = 0$ and with $k^+ \to \frac{2\pi}{L} m$, etc.:

$$\Sigma(0) = V_3(\mu^2) = -\frac{1}{\mu^2} \sum_{m=1}^{K-2} \frac{1}{m} \sum_{n=1}^{K-m-1} \frac{1}{n(K-m-n)\left[\frac{1}{m} + \frac{1}{n} + \frac{1}{K-m-n}\right]}. \tag{29}$$

Numerical values:

$$\begin{array}{ccccc} K = 32 & K = 64 & K = 128 & K = 512 & K = 2048 \\ V = 1.921 & V = 2.099 & V = 2.205 & V = 2.301 & V = 2.331 \end{array} \tag{30}$$

Smooth approach to $p = 0$ $(K = 512)$:

$$\begin{array}{cccc} p^2 = 10^{-2} & p^2 = 10^{-4} & p^2 = 10^{-6} & p^2 = 0 \\ V = 3.267 & V = 2.307 & V = 2.301 & V = 2.302 \end{array} \quad (31)$$

Explanation: for some finite p^-, $p^- = \frac{\mu^2}{p^+}$ approaching $\mu^2 \to 0$ implies p^+ approaches 0 as well

Convergence for the ϕ^4 loop slower, but reliable:

$$V_4(\mu^2) = -\frac{1}{\mu^2} \sum_{l=1}^{K-3} \frac{1}{l} \sum_{m=1}^{K-l-2} \frac{1}{m} \sum_{n=1}^{K-l-l-1} \frac{1}{n(K-l-m-n)\left[\frac{1}{l} + \frac{1}{m} + \frac{1}{n} + \frac{1}{K-l-m-n}\right]} \quad (32)$$

$V_4 = 6.798, \quad 7.795, \quad 7.967$ for $K = 128, \quad 512, \quad 800$, approaching the continuum value $V_4 = 8.414....$

CONCLUSIONS

- vacuum diagrams in the $\phi^3(1+1)$ and $\phi^4(1+1)$ models obtained as $p = 0$ (external momentum) limit of the corresponding self-energy diagrams

- works also in a final volume with (A)PBC (DLCQ) $\Rightarrow$ not effect of the zero modes

- generalization to e.g. Yukawa theory and to (3+1)-dimensional case straightforward

- expected to work also for the generalized tadpoles - to be checked

Simple tadpole and a generalized tadpole in ϕ^4 model