Non-Abelian Vortex in Four Dimensions as a Critical Superstring

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1 Introduction

Confinement is not just ONE problem. It is TWO problems

- Understand the nature of confining strings

  What do we know?

  - Lattice
  - Supersymmetry: Seiberg-Witten solution of $\mathcal{N} = 2$ QCD. Abelian
  - Non-Abelian generalizations?

    Non-Abelian vortex strings

    Quarks condense $\Rightarrow$ monopoles are confined

- Quantize confining string outside critical dimension

  What do we know??????

  Shifman and Yung, 2015: Non-Abelian vortex in $\mathcal{N} = 2$ supersymmetric QCD can behave as a critical superstring

Non-Abelian vortex strings

Non-Abelian strings were found in $\mathcal{N}=2$ U(N) QCD

Hanany, Tong 2003

Auzzi, Bolognesi, Evslin, Konishi, Yung 2003

Shifman Yung 2004

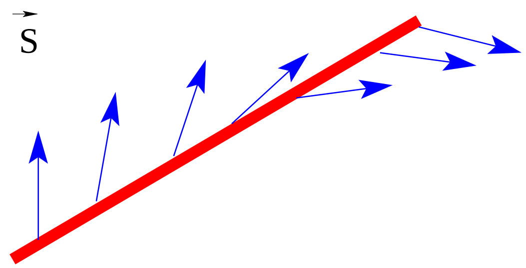
Hanany Tong 2004

Z_N Abelian string: Flux directed in the Cartan subalgebra, say for $SO(3)=SU(2)/Z_2$

$$flux \sim \tau_3$$

Non-Abelian string : **Orientational zero modes**

Rotation of color flux inside SU(N).



Idea:

Non-Abelian string has more moduli then Abrikosov-Nielsen-Olesen (ANO) string.

It has translational + orientaional moduli

We can fulfill the criticality condition: In $\mathcal{N}=2$ QCD with U$(N=2)$ gauge group and $N_f=4$ quark flavors.

- The solitonic non-Abelian vortex have six orientational moduli, which, together with four translational moduli, form a ten-dimensional space.
- For $N_f=2N$ 2D world sheet theory on the string is conformal.

Most of solitonic strings are ”thick”.

Transverse size $= \frac{1}{m}$, where m is the typical mass of bulk excitations.

$$S_{2D} = T \int d^2\sigma \left\{ (\mathrm{LE}\ \ \sigma\mathrm{-model}) + O\left(\frac{\partial^n}{m^n}\right) \right\}$$

where T is string tension

Polchinski-Strominger, 1991: Without higher derivative terms

the world sheet theory is not UV complete

Given that for non-Abelian vortex low energy world sheet theory is critical we conjecture that

Thin string regime

$$T \ll m^2$$

is actually satisfied at strong coupling $g_c^2 \sim 1$.

$$m(g) \to \infty, \qquad g^2 \to g_c^2$$

Higher derivative corrections can be ignored

2 Non-Abelian vortex strings

Bulk theory: 4D $\mathcal{N} = 2$ QCD with Fayet-Iliopoulos term.

For $U(N)$ gauge group in the bulk we have 2D $CP(N-1)$ model on the string

CP$(N-1)$ == U(1) gauge theory in the strong coupling limit

$$S_{\mathrm{CP}(N-1)} = \int d^2x \left\{ \left| \nabla_\alpha n^P \right|^2 + \frac{e^2}{2} \left(|n^P|^2 - \beta \right)^2 \right\} ,$$

where n^P are complex fields $P = 1, ..., N$,

Condition

$$|n^P|^2 = \beta \approx \frac{4\pi}{g^2},$$

imposed in the limit $e^2 \to \infty$

More flavors $\Rightarrow$ semilocal non-Abelian string

The orientational moduli described by a complex vector n^P (here $P = 1, ..., N$),

$\tilde{N} = (N_f - N)$ size moduli are parametrized by a complex vector ρ^K ($K = N + 1, ..., N_f$).

The effective two-dimensional theory is the $\mathcal{N} = (2, 2)$ weighted CP model

$$S_{\rm WCP} = \int d^2x \left\{ \left|\nabla_\alpha n^P\right|^2 + \left|\tilde{\nabla}_\alpha \rho^K\right|^2 + \frac{e^2}{2}\left(|n^P|^2 - |\rho^K|^2 - \beta\right)^2 \right\},$$

$$P = 1, ..., N\,, \qquad K = N + 1, ..., N_f\,.$$

The fields n^P and ρ^K have charges +1 and -1 with respect to the auxiliary U(1) gauge field

$e^2 \to \infty$

Global group

$$SU(N) \times SU(\tilde{N}) \times U(1)$$

3 From non-Abelian vortices to critical strings

String theory

$$S = \frac{T}{2}\int d^2\sigma\sqrt{h}\, h^{\alpha\beta}\partial_\alpha x^\mu\,\partial_\beta x_\mu$$

$$+ \int d^2\sigma\sqrt{h}\,\Big\{h^{\alpha\beta}\left(\tilde{\nabla}_\alpha\bar{n}_P\,\nabla_\beta\,n^P + \nabla_\alpha\bar{\rho}_K\,\tilde{\nabla}_\beta\,\rho^K\right)$$

$$+ \frac{e^2}{2}\left(|n^P|^2 - |\rho^K|^2 - \beta\right)^2\Big\} + \text{fermions}\,,$$

where $h^{\alpha\beta}$ is the world sheet metric. It is independent variable in the Polyakov formulation.

Criticality conditions

- Conformal invariance

$$b_{WCP} = N - \tilde{N} = 0 \Rightarrow N = \tilde{N}, \quad N_f = 2N$$

- Critical dimension =10

  Number of orientational + size degrees of freedom
  $= 2(N + \tilde{N} - 1) = 2(2N - 1)$

$$4 + 2(2N - 1) = 4 + 6 = 10, \qquad \text{for } N = 2$$

Our string is BPS so we have $\mathcal{N} = (2, 2)$ supersymmetry on the world sheet.

For these values of N and $\tilde{N}$ the target space of the weighted $CP(2, 2)$ model is a non-compact Calabi-Yau manifold studied by Candelas, Witten and Vafa, namely

conifold.

Strings in the $U(N)$ theories are stable; they cannot be broken. Thus, we deal with the closed string.

For closed string moving on Calabi-Yau manifold $\mathcal{N} = (2, 2)$ world sheet supersymmetry ensures $\mathcal{N} = 2$ supersymmetry in 4D.

This is expected since we started with 4D QCD with $\mathcal{N} = 2$ supersymmetry.

Type IIB string is a chiral theory and breaks parity while Type IIA string theory is left-right symmetric and conserves parity.

Our bulk theory conserves parity $\Rightarrow$ we have Type IIA superstring

We conjectured that the string becomes thin $m \to \infty$ at $g^2 \to g_c^2 \sim 1.$

$$g^2 \quad \Longleftrightarrow \quad \beta$$

4D coupling 2D coupling

It is natural to expect that

$$g_c^2 \quad \Longleftrightarrow \quad \beta = 0$$

D-term condition in weighted CP(2,2) model

$$|n^P|^2 - |\rho^K|^2 = \beta, \qquad P = 1, 2, \qquad K = 1, 2$$

At $\beta = 0$ conifold develops conical singularity.

4 4D massless states

Our goal:

Study states of closed string propagating on

$$R_4 \times Y_6, \qquad Y_6 = \text{conifold}$$

and interpret them as hadrons in 4D $\mathcal{N} = 2$ QCD.

Massless states = Deformations of 10D metric preserving Ricci flatness

Massless 4D graviton

Constant wave functions over conifold

Non-normalizable on non-compact Y_6.

No 4D graviton == good news!

We do not have gravity in our 4D $\mathcal{N} = 2$ QCD

Kahler form deformations

Kahler form deformations = variations of 2D coupling β

D-term condition in weighted CP(2,2) model

$$|n^P|^2 - |\rho^K|^2 = \beta, \qquad P = 1, 2, \qquad K = 1, 2$$

Resolved conifold

β - non-normalizable mode

5 Deformation of the complex structure

D-term condition

$$|n^P|^2 - |\rho^K|^2 = \beta, \qquad P = 1, 2, \qquad K = 1, 2$$

Construct U(1) gauge invariant ”mesonic ” variables”

$$w^{PK} = n^P \rho^K.$$

$$\det w^{PK} = 0$$

Take $\beta = 0$

Complex structure deformation $\Rightarrow$ Deformed conifold

$$\det w^{PK} = b$$

b – complex modulos

The effective action for $b(x)$ is

$$S(\beta) = T \int d^4x \, h_b (\partial_\mu b)^2,$$

where

$$h_b = \int d^6 y \sqrt{g} g^{li} \left(\frac{\partial}{\partial b} g_{ij} \right) g^{jk} \left(\frac{\partial}{\partial \bar{b}} g_{kl} \right)$$

Using explicit Calabi-Yau metric on deformed conifold we get

$$h_b = (4\pi)^3 \, \frac{4}{3} \, \log \frac{T^2 L^4}{|b|}$$

For Type IIA string b should be a part of hypermultiplet.

6 Non-Abelian vortex and Little String Theory

For $\beta = 0$ supergravity approximation does not work.

Still can be used for massless states = chiral primary operators (4D BPS states)

Protected

Consider massive states

Ghoshal, Vafa, 1995; Giveon Kutasov 1999

Critical string on a conifold is equivalent to non-critical $c = 1$ string

$$\mathcal{R}^4 \times \mathcal{R}_\phi \times S^1,$$

$\mathcal{R}_\phi$ is a real line associated with the Liouville field ϕ and the theory has a linear in ϕ dilaton, such that string coupling is given by

$$g_s = e^{-\frac{Q}{2}\phi}\,.$$

Aharony, Berkooz, Kutasov, Seiberg, 1994

String theories with this behavior of the dilaton are holographic – ”Little String Theories”

Non-trivial dynamics is localized on the $\mathcal{R}^4$ boundary

This is exactly what we want!

We expect that LST in our case is 4D $\mathcal{N}=2$ supersymmetric QCD at the self-dual value of the gauge coupling $g^2=4\pi$ (in the hadronic description)

$$T_{--}=-\frac{1}{2}\left[(\partial_z\phi)^2+Q\,\partial_z^2\phi+(\partial_z Y)^2\right]$$

$$Y\sim Y+2\pi Q \qquad Q=\sqrt{2}, \qquad c_{\phi+Y}^{SUSY}=3+3Q^2=9$$

Liouville interaction

$$\delta L=b\int d^2\theta\, e^{-\frac{\phi+iY}{Q}}$$

Mirror description: $SL(2,R)/U(1)$ WZNW model at level $k = 1$.

Bosonic part is 2D Witten's black hole with target space forming semi-infinite cigar.

Lioville field ϕ – motion along the cigar.

The spectrum of primary operators was computed exactly.

Dixon, Peskin, Lykken, 1989; Mukhi, Vafa, 1993; Evans, Gaberdiel, Perry, 1998

$$V_{j,m} \approx \exp\left(\sqrt{2} j\phi + i\sqrt{2} mY\right), \qquad \phi \to \infty$$

- Normalizable states – discrete series with $j \le -\frac{1}{2}$
- No negative norm states

$$j = -\frac{1}{2}, \qquad m = \pm\left\{\frac{1}{2}, \frac{3}{2}, \ldots\right\}$$

$$j = -1, \qquad m = \pm\{1, 2, \ldots\}$$

10D "tachyon"

$$V^S_{j,m}(p_\mu) = e^{-\varphi}\, e^{ip_\mu x^\mu}\, V_{j,m}\,, \qquad j = -\frac{1}{2}, \qquad m = \pm\left\{\frac{1}{2}, \frac{3}{2}, \dots\right\}$$

$$\frac{(M^S)^2}{8\pi T} = -\frac{p_\mu p^\mu}{8\pi T} = m^2 - \frac{1}{2} - j(j+1) = m^2 - \frac{1}{4} = 0, 2, 6, \dots$$

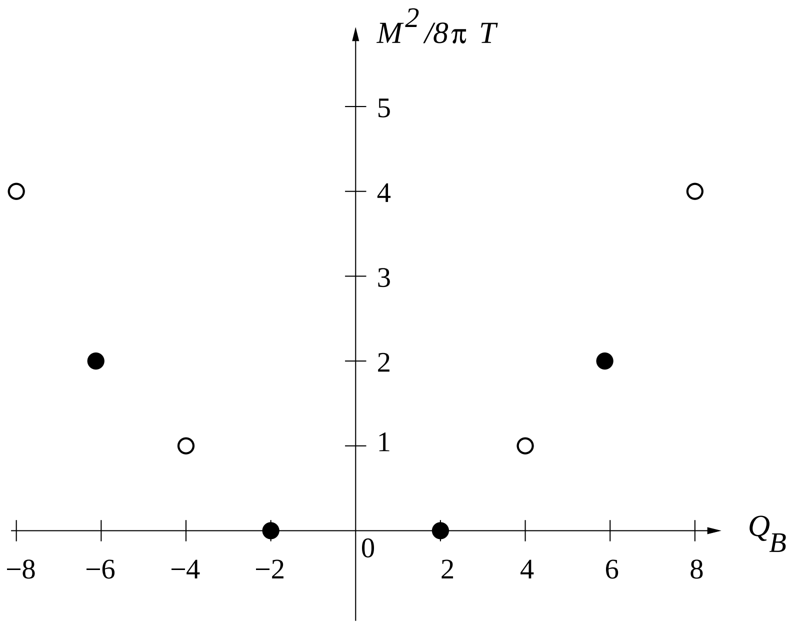
Massless state at $m = \pm\frac{1}{2}$ – b-state

Spin-2 states

$$V^G_{j,m}(p_\mu) = \xi_{\mu\nu}\psi^\mu_L\psi^\nu_R\, e^{-\varphi}\, e^{ip_\mu x^\mu}\, V_{j,m}, \qquad j = -1, \qquad m = \pm\{1,\, 2, \dots\}$$

$$\frac{(M^G)^2}{8\pi T} = m^2 = 1, 4, 9, \dots$$

No massless graviton



Global group of the 4D QCD:

$$SU(2) \times SU(2) \times U(1)$$

U(1) - ”baryonic” symmetry.

$$Q_B = 4m$$

7 Superultiplet structure

Lowest states:

- Massless state b $\qquad j = -\frac{1}{2}, m = \pm\frac{1}{2}$

  Short BPS multiplet

$$\text{Hypermultiplet} = 4_{scalar} + \text{fermions}$$

- $j = -\frac{1}{2}, m = \pm\frac{3}{2}$

$$\frac{\left(M_{j=-\frac{1}{2}, m=\pm 3/2}\right)^2}{8\pi T} = 2$$

  Two long non-BPS vector supermultiplets

$$(\mathcal{N} = 2\ )_{\text{vector}} = 1_{\text{vector}} + 5_{\text{scalar}} + \text{fermions}$$

- $j = -1, m = \pm 1$

$$\frac{(M_{j=-1,m=\pm 1})^2}{8\pi T} = 1$$

$$(j=-1)\,\mathrm{states} = 2 \times (\mathcal{N}=2\ )_{\mathrm{spin-2}} + 4 \times (\mathcal{N}=2\ )_{\mathrm{vector}}$$

where

$$(\mathcal{N}=2\ )_{\mathrm{spin-2}} = 1_{\mathrm{spin-2}} + 6_{\mathrm{vector}} + 1_{\mathrm{scalar}} + \mathrm{fermions}$$

8 Monopole necklace baryons

Strings in the $U(N)$ theories are stable; they cannot be broken. Thus, we deal with the closed string.

Quarks are condensed in 4D theory. Therefore, monopoles are confined.

In $U(N)$ gauge theories the confined monopoles are junctions of two non-Abelian vortex strings.

a b

Monopole-antimonopole meson Monopole-monopole baryon

9 Conclusions

- In $\mathcal{N} = 2$ supersymmetric QCD with gauge group U(2) and $N_f = 4$ quark flavors non-Abelian BPS vortex behaves as a critical superstring.
- Massless closed string state b associated with deformations of the complex structure of the conifold == monopole-monopole baryon.
- Successful tests of our gauge-string duality:
  - $\mathcal{N} = 2$ supersymmetry in 4D QCD
  - Absence of graviton and unwanted vector fields.
- Spectrum of lowest massive baryons is calculated using ”Little String Theory” description

  We calculate hadron spectrum from first principles!

Higher derivative terms at weak coupling, $g \ll 1$

$$O\left(\frac{\partial^n}{m^n}\right), \qquad m \sim g\sqrt{T}$$

At $J \sim 1 \qquad \partial \to \sqrt{T}$

Thus higher derivative terms

$$\to \left(\frac{T}{m^2}\right)^n$$

blow up at weak coupling!

Polyakov: string surface become ”crumpled”.

4D interpretation: String grows short and thick.

$$L^2 \sim \frac{J}{T} \lesssim \frac{1}{m}, \qquad \text{for } J \sim 1$$

There is self-duality in 4D bulk theory

$$\tau \to \tau_D = -\frac{1}{\tau}, \qquad \tau = \frac{4\pi i}{g^2} + \frac{\theta_{4D}}{2\pi},$$

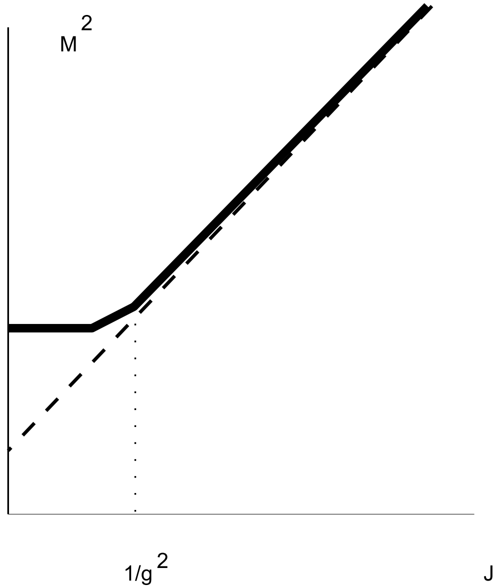
We conjectured that the string becomes thin at $g^2 \to g_c^2 \sim 1$.

It is natural to expect that $g_c^2 = 4\pi$ = self-dual point.

$$m^2 \to T \times \begin{cases} g^2, & g^2 \ll 1 \\ \infty, & g^2 \to 4\pi \\ 16\pi^2/g^2, & g^2 \gg 1 \end{cases},$$

In 2D theory on the string self-dual point is $\beta = 0$

Conifold develops conical singularity.



QUESTION:

Can we find **any** example of a 4D field theory which supports thin vortex strings?

Non-Abelian vortex in $\mathcal{N} = 2$ QCD with U(2) gauge group and $N_f = 4$ flavors is critical.

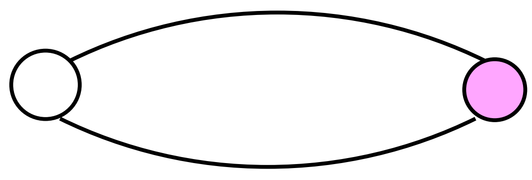
$\mathcal{N}=2$ supersymmetric QCD with gauge group $U(N)$ and N_f quark flavors

(Scalar) quarks condense $\Rightarrow$ monopoles are confined

Strings in the $U(N)$ theories are stable; they cannot be broken.

In $U(N)$ gauge theories the confined monopoles are junctions of two non-Abelian vortex strings.

Example



Monopole-antimonopole meson

Constituent quark = monopole

Physical nature of non-normalizable modes

Gukov, Vafa, Witten 1999: Non-normalizable moduli = coupling constants in 4D

- 4D metric do not fluctuate. It is fixed to be flat. ”Coupling constants.”
- 2D coupling β is related to 4D coupling g^2. Fixed. Non-dynamical.

Another option:

Large $y_i \Rightarrow$ large n^P and ρ^K

Non-normalizable modes are not localized on the string.

Unstable states. Decay into massless perturbative states.

Higgs branch: $\dim\mathcal{H} = 4N\tilde{N} = 16$.

Strong coupling

Global group of the 4D QCD:

$$SU(2) \times SU(2) \times U(1)$$

U(1) - "baryonic" symmetry.

b-hypermultiplet: $(1,\ 1,\ 2)$

Logarithmically divergent norm == Marginal stability at $\beta = 0$

b-state can decay into massless bi-fundamental (screened) quarks living on the Higgs branch.