q-Virasoro constraints from q-difference operators

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work in progress with
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Motivation

- QFT is hard.

  divergencies, perturbation theory is asymptotic, renormalization issues

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- Arise naturally (e.g. $\mathcal{N} = 2$ YM-CS $U(N)$ theories):

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- Kazakov-Migdal model for QCD

$$Z=\int\prod_x d\phi_x\prod_{xy}[dU_{xy}]e^{S[\phi,U]},$$

$$S[\phi,U]=-\sum_x N\mathrm{tr}V(\phi_x)+\sum_{xy}N\mathrm{tr}\phi_x U_{xy}\phi_y U_{xy}^\dagger$$

Hermitean Gaussian matrix model

Partition function, normalized averages

$$Z=\int_{H_N}\prod_{i=1}^{N}d\phi_{ii}\prod_{i<j}d\phi_{ij}d\bar{\phi}_{ij}e^{-\frac{1}{2}\mathrm{tr}\phi^2},\quad \langle f(\phi)\rangle=\frac{\int[d\phi]e^{-\frac{1}{2}\mathrm{tr}\phi^2}f(\phi)}{Z}$$

The correlators

$$C_{k_1,\ldots,k_n}=\left\langle \mathrm{tr}\phi^{k_1}\ldots\mathrm{tr}\phi^{k_n}\right\rangle$$

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Can diagonalize $\phi = U \Lambda U^\dagger$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$

$$Z = \int_{-\infty}^{\infty} d\lambda_1 \ldots d\lambda_n \prod_{i \neq j} (\lambda_i - \lambda_j)^2 \exp\left(-\frac{1}{2} \sum_{i=1}^{N} \lambda_i^2 \right)$$

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Can consider formal generating function

$$Z(\vec{t}) = \left\langle \exp\left(t_0 N + \sum_{k=1}^{\infty} t_k \sum_{i=1}^{N} \lambda_i^k\right) \right\rangle, \quad C_{k_1,\ldots,k_n} = \frac{\partial^n}{\partial t_{k_1} \ldots \partial t_{k_n}} \bigg|_{\vec{t}=0} Z(\vec{t})$$

Ward identities, Virasoro constraints via full derivatives

Insert $\sum_{i=1}^{N} \frac{\partial}{\partial \lambda_i} \lambda_i^{n+1}, n = -1, 0, 1, \ldots$ and use $\int d\lambda \frac{\partial}{\partial \lambda}(\ldots) = 0$

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$$\sum_i \lambda_i^{n+1} \frac{\partial}{\partial \lambda_i} \exp(-\frac{1}{2} \sum \lambda_i^2) = \exp(-\frac{1}{2} \sum \lambda_i^2)(-1) \sum_i \lambda_i^{n+2}$$

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$$\sum_i \lambda_i^{n+1}\frac{\partial}{\partial\lambda_i}\exp\left(t_0 N + \sum_{k=1}^{\infty} t_k\sum_i\lambda_i^k\right) = \exp(\ldots)\left(\sum_{k=1}^{\infty} t_k k\sum_i\lambda_i^{k+n}\right)$$

Can rewrite as a system of PDEs on $Z(\vec{t})$:

$$\left(\sum_{a=0}^{n}\frac{\partial^2}{\partial t_a\partial t_{n-a}} - \frac{\partial}{\partial t_{n+2}} + \sum_{k=0}^{\infty} k t_k\frac{\partial}{\partial t_{k+n}}\right)Z(\vec{t}) = 0, \quad L_n Z(\vec{t}) = 0$$

The Virasoro algebra

L_n commute like

$$[L_n, L_m] = (n-m)L_{n+m}$$

and so do $l_n = \sum_i \frac{\partial}{\partial\lambda_i}\lambda_i^{n+1}$

$$[l_n, l_m] = (n-m)l_{n+m}$$

Virasoro constraints on the level of correlators

$$\left(\sum_{a=0}^{n} \frac{\partial^2}{\partial t_a \partial t_{n-a}} - \frac{\partial}{\partial t_{n+2}} + \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} \right) Z(\vec{t}) = 0$$

imply that

$$C_{n+2,k_1,\ldots k_m} = \sum_{a=0}^{n} C_{a,n-a,k_1,\ldots k_m} + \sum_{j=1}^{m} k_j C_{k_1,.,k_j+n,.,k_m}$$

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imply that

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$+$ initial conditions

$$C_{0,\dots} = N C_{\dots}, \quad C_{\emptyset} = 1, \quad C_1 = 0$$

The (q,t)-deformed Gaussian model

We want to preserve a simple property

$$< \lambda^p >_{N=1} = (p-1)!!\delta_{p|2} \underset{(q,t)}{\longrightarrow} < \lambda^p >_{N=1} = [p-1]_q!!\delta_{p|2}, \quad [n]_q = \frac{(1-q^n)}{(1-q)}$$

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We change all the ingredients accordingly

$$\prod_{i\neq j}(\lambda_i - \lambda_j) \underset{\substack{(q,t) \\ t=q^\beta}}{\longrightarrow} \prod_{i\neq j}\prod_{m=0}^{\beta-1}(\lambda_i - q^m\lambda_j) \underset{\beta\notin\mathbb{Z}}{\longrightarrow} \prod_{i\neq j}\frac{\prod\limits_{m=0}^{\infty}\left(1-q^m\frac{\lambda_i}{\lambda_j}\right)}{\prod\limits_{m=0}^{\infty}\left(1-q^{\beta+m}\frac{\lambda_i}{\lambda_j}\right)}\prod_i \lambda_i^{\beta(N-1)}$$

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$$\exp\left(-\frac{1}{2}\lambda^2\right) \underset{(q,t)}{\longrightarrow} e_q\left(\frac{\lambda^2}{[2]_q}\right) = \prod_{m=0}^{\infty}\left(1-q^{2m+2}(1-q)\lambda^2\right)$$

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$$\int d\lambda \underset{(q,t)}{\longrightarrow} \text{Jackson integral} \int d_q\lambda f(\lambda) = (1-q)\sum_{n=0}^{\infty}\nu q^n\left[f(\nu q^n)+f(-\nu q^n)\right], \; \nu = \frac{1}{\sqrt{1-q}}$$

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$$\exp\left(\sum t_k\lambda^k\right) \text{ does not change}$$

q-difference operators

The full derivative vanishing

$$\int d\lambda \frac{\partial}{\partial \lambda}(\ldots) = 0 \underset{(q,t)}{\longrightarrow} \int d_q\lambda \frac{1}{\lambda}\left(\lambda^{\dagger} - 1\right)(\ldots) = 0, \text{ where } \lambda^{\dagger} f(\lambda) = f\left(q^{-1}\lambda\right)$$

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The full q-difference operators

$$\sum_i \frac{\partial}{\partial\lambda_i}\lambda_i^{n+1} \underset{(q,t)}{\longrightarrow} \sum_i \frac{1}{\lambda_i}\left(\lambda_i^\dagger - 1\right)\lambda_i^{n+1}\prod_{j\neq i}\frac{\lambda_j - t\lambda_i}{\lambda_j - \lambda_i}, \quad n = -1, 0, 1, \ldots$$

Some details about the derivation

$$\lambda_i^\dagger \Delta = \Delta \prod_{i\neq j} \frac{\left(1-\frac{\lambda_i}{q\lambda_j}\right)}{\left(1-\frac{t\lambda_i}{q\lambda_j}\right)} \prod_{k\neq i} \frac{\left(1-t\frac{\lambda_k}{\lambda_i}\right)}{\left(1-\frac{\lambda_k}{\lambda_i}\right)}$$

$$\lambda_i^\dagger e_q\left(\frac{\lambda^2}{[2]_q}\right) = e_q\left(\frac{\lambda^2}{[2]_q}\right)\left(1-(1-q)\lambda_i^2\right)$$

$$\lambda_i^\dagger \exp\left(\sum_{k=1}^{\infty} t_k \sum_{j=1}^{N} \lambda_j^k\right) = \exp\left(\sum_{k=1}^{\infty} t_k \sum_{j=1}^{N} \lambda_j^k\right) \exp\left(\sum_{k=1}^{\infty} t_k \lambda_i^k (q^{-k}-1)\right)$$

Rewrite extra pieces as a sum over residues

$$\oint_{\omega=\lambda_i/q} \frac{d\omega}{\omega}\omega^{n+1}\left(1-q^2(1-q)\omega^2\right)\prod_{i=1}^{N}\frac{q\omega-t\lambda_i}{q\omega-\lambda_i}\exp\left(\sum_{k=1}^{\infty} t_k\omega^k(1-q^k)\right)$$

Move integration contour to the one around 0 and ∞

Calculate the residue at ∞.

Residue at 0 extracts a certain coefficient of Laurent expansion at 0.

Ward identities generating equation

After some massaging, the following equations can be obtained

$$
\begin{aligned}
& tQ^{-1}\left(1-q^2(1-q)y^2\right)\exp\left(\sum_{k=1}^{\infty}(1-q^k)t_k y^k\right) Z\left(t_k \to t_k + \frac{(1-t^k)}{kq^k}\frac{1}{y^k}\right) \\
& + qQZ\left(t_k \to t_k + \frac{(1-t^{-k})}{k}\frac{1}{y^k}\right) \\
& - (q+t)Z\left(t_k\right) - yt(1-q)t_1 Z\left(t_k\right) = \sum_{m=0}^{\infty} c_m y^{m+2}
\end{aligned}
$$

from which we can select a specific degree components in y and $\vec{t}$

$$
\begin{aligned}
& tQ^{-1}q^2(1-q)s_{m+2}\left(p_k = \frac{(1-t^k)}{q^k}\frac{\partial}{\partial t_k}\right) Z_{d+2} \\
& = tQ^{-1}s_m\left(p_k = \frac{(1-t^k)}{q^k}\frac{\partial}{\partial t_k}\right) Z_d \\
& \quad + qQ\sum_{p=0}^{d-m} s_p\left(p_k = -(1-q^k)kt_k\right) s_{p+m}\left(p_k = (1-t^{-k})\frac{\partial}{\partial t_k}\right) Z_d \\
& \quad - \delta_{m,0}(q+t)Z_d + \delta_{m,-1}q(1-q)t_1 Z_d,
\end{aligned}
$$

Ward identities in terms of correlators

And using explicit formula for the Schur polynomials $s_m(\vec{p})$ we can write for correlators

$$
\begin{aligned}
& tQ^{-1}q^2(1-q)\frac{1}{\lambda_\bullet}\frac{(1-t^{\lambda_\bullet})}{q^{\lambda_\bullet}}C_{\lambda_1\ldots\lambda_\bullet} \\
&= -tQ^{-1}q^2(1-q)\sum_{\substack{|\vec{\mu}|=\lambda_\bullet \\ l(\vec{\mu})\geq 2}}\frac{1}{l(\vec{\mu})!}\left(\prod_{a\in\vec{\mu}}\frac{(1-t^a)}{q^a a}\right)C_{\lambda_1\ldots\lambda_{\bullet-1}\mu_1\ldots\mu_\bullet} \\
&\quad -\delta_{\lambda_\bullet,2}(q+t)C_{\lambda_1\ldots\lambda_{\bullet-1}}+\delta_{\lambda_\bullet,1}((\#_\lambda 1)-1)q(1-q)C_{\lambda_1\ldots\lambda_{\bullet-2}} \\
&\quad + tQ^{-1}\sum_{|\vec{\mu}|=\lambda_\bullet-2}\frac{1}{l(\vec{\mu})!}\left(\prod_{a\in\vec{\mu}}\frac{(1-t^a)}{q^a a}\right)C_{\lambda_1\ldots\lambda_{\bullet-1}\mu_1\ldots\mu_\bullet} \\
&\quad + qQ\sum_{\nu\subseteq\lambda\backslash\lambda_\bullet}\left(\prod_{a\in\nu}(-1)(1-q^a)\right)\sum_{|\vec{\mu}|=|\nu|+\lambda_\bullet-2}\frac{1}{l(\vec{\mu})!}\left(\prod_{a\in\vec{\mu}}\frac{(1-t^{-a})}{a}\right)C_{\lambda\backslash\{\lambda_\bullet,\nu\}\vec{\mu}}
\end{aligned}
$$

Relation to the q-Virasoro

We can rewrite Ward identities in the form

$$T(y)Z(\vec{t}) = \left(p^{1/2} + p^{-1/2}\right) Z(\vec{t}) + 0 \cdot y + \sum_{m=2}^{\infty} c_m y^m$$

$T(y) = \sum_{n=-\infty}^{\infty} T_n z^{-n}$ is the q-Virasoro current $(p = qt^{-1})$

$$\begin{aligned} T(z) =& p^{1/2} \exp\left(-\sum_{n=1}^{\infty} \frac{1-t^n}{1+p^n} \frac{a_{-n}}{n} z^n t^{-n} p^{-n/2}\right) \exp\left(-\sum_{n=1}^{\infty} (1-t^n) \frac{a_n}{n} z^{-n} p^{n/2}\right) q^{\beta a_0} \\ & p^{-1/2} \exp\left(\sum_{n=1}^{\infty} \frac{1-t^n}{1+p^n} \frac{a_{-n}}{n} z^n t^{-n} p^{n/2}\right) \exp\left(\sum_{n=1}^{\infty} (1-t^n) \frac{a_n}{n} z^{-n} p^{-n/2}\right) q^{-\beta a_0} \end{aligned}$$

and one identifies

$$a_n = q^{-n/2} t^{-n/2} \frac{\partial}{\partial t_n} \qquad a_{-n} = q^{n/2} t^{n/2} n \frac{(1-q^{|n|})}{(1-t^{|n|})} t_n$$

Applications and open questions

- Observables in $\mathcal{N} = 2$ YM-CS theory on S^3

$$\text{SUSY Wilson lines} \leftrightarrow \langle s_\lambda \rangle$$

- Averages of Macdonald polynomials are nice

$$\langle \mathcal{M}_\lambda \rangle = \mathcal{M}_\lambda \left(p_k = (1 + (-1)^k) \frac{(1-q)^{k/2}}{(1-t^k)} \right) \prod_{(i,j)\in\lambda} \frac{1 - t^{N+1-i} q^{j-1}}{1-q}$$

- Loop equations, topological recursion and Givental formalism for CohFTs?
- $\hat{W}$-operator (cut-and-join) representation and connections to enumerative geometry?

$$Z = \exp\left(\hat{W}_{-2}\right) \exp\left(N t_0\right),$$

$$\hat{W}_{-2} = \sum_{a,b=0}^{\infty} \left(a b t_a t_b \frac{\partial}{\partial t_{a+b-2}} + (a+b-2) t_{a+b+2} \frac{\partial^2}{\partial t_a \partial t_b} \right)$$

To summarize:

- (q,t)-Gaussian matrix model has a large set of symmetries that form q-Virasoro algebra;
- These symmetries can be easily derived using insertions of q-difference operators;
- There are many exciting open questions to which this q-Virasoro symmetry could be applied.

Thank you for your attention!