On vibrational modes of Q-balls

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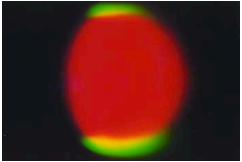
Overview

Overview of Q-balls

- Q-balls are a class of non-topological solitons existing in theories of complex scalar field possessing (global) $U(1)$-symmetry (**Rosen, G.'68; Coleman, S.'85**)
- They are well-studied objects with numerous applications in astrophysics and cosmology, condensed matter physics, even non-linear optics.
- They allow for simple (sometimes analytical) treatment (**Theodorakis, S'00**)
- They can serve as prototypes for more complicated objects arising in realistic setting (e.g. boson stars).



Setup

- We will be interested in the models of one complex scalar field in 3+1 dimensions with a potential of a special form,

General Lagrangian

$$\mathcal{L} = |\partial_\mu \phi|^2 - V(|\phi|)$$

The ansatz, energy and charge of a Q-ball

$$\phi = f(r) e^{i\omega t} \,, \qquad E = \int d^3x \, ((\vec{\nabla} f)^2 - \omega^2 f^2 + V(f)) \,, \qquad Q = 2\omega \int d^3x \, f^2$$

We will study small perturbations on top of these configurations.

- The potential will be chosen so that to allow for analytical treatment of both the solitons and their perturbations.
- Note that the problem of finding a spectrum of bound states of a Q-ball is *not* an eigenvalue problem for an Hermitian operator.

Q-balls in the flat potential

Consider a potential consisting of two parabolic branches joined at some point $|\varphi| = v$. Require the presence of a flat direction,

Parabolic potential with the flat direction

$$V(|\phi|) = m^2|\phi|^2\theta\left(1 - \frac{|\phi|^2}{v^2}\right) + m^2v^2\theta\left(\frac{|\phi|^2}{v^2} - 1\right)$$

The set of Q-balls split on two branches. One of them (with $\omega < \omega_c$) contains *classically stable* solutions. Another (with $\omega > \omega_c$) corresponds to unstable "Q-clouds" (**Alford, M. G.'88**)

The critical frequency $\omega_c \approx 0.960m$ corresponds to the soliton with the minimal possible energy and charge.

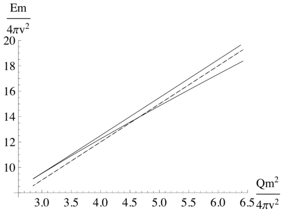


Figure: $E(Q)$ for the Q-balls in the flat parabolic potential (**Gulamov, I. E., Nugaev, E. Ya. and Smolyakov, M. N.'13**)

Perturbations of Q-balls in the flat potential

An appropriate ansatz governing the dynamics of small oscillations on top of the classically stable Q-balls reads as follows (**M. N. Smolyakov'18**)

Perturbation ansatz

$$\phi = \phi_0 + \psi e^{i\omega t} \ , \qquad \psi(\vec{x}, t) = (\psi_1^{(l)}(r) e^{i\gamma t} + \psi_2^{(l)}(r) e^{-i\gamma t}) Y_{l,m}(\theta, \varphi)$$

where the parameter γ is taken to be real and positive, $\psi_1^{(l)}$, $\psi_2^{(l)}$ are real functions of the radial coordinate and $Y_{l,m}$ are spherical harmonics.
Substituting this into the linearized equations of motion, one gets

Equations for perturbations

$$\left(\Delta_r - \frac{l(l+1)}{r^2} + (\omega+\gamma)^2 - g(r)\right)\psi_1^{(l)}(r) - h(r)\psi_2^{(l)*}(r) = 0$$
$$\left(\Delta_r - \frac{l(l+1)}{r^2} + (\omega-\gamma)^2 - g(r)\right)\psi_2^{(l)}(r) - h(r)\psi_1^{(l)*}(r) = 0$$

The functions g and h are determined by the potential. In our case

$$h(r) = -\frac{m^2}{2}\delta\left(\frac{f(r)}{v} - 1\right) \ , \qquad g(r) = m^2\theta\left(1 - \frac{f^2(r)}{v^2}\right) + h(r)$$

Hence, equations are disentangled everywhere except the single point R such that $f(R) = v$.

Perturbations of Q-balls in the flat potential

To study bound states, one imposes

Boundary conditions

$$\psi_{1,2}^{(l)}(\infty) = 0 \, , \qquad \partial_r \psi_{1,2}^{(l)}\Big|_{r=0} = 0$$

and also $\gamma + \omega < m$.

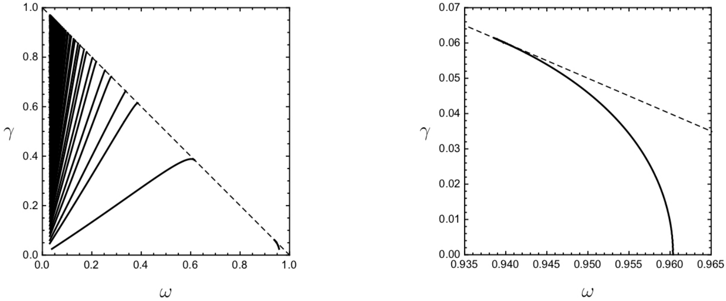


Figure: The discrete spectrum of linear perturbations of classically stable Q-balls in the flat potential, at $l = 0$. All quantities are normalized to the parameter m.

Perturbations of Q-balls in the flat potential

Features of the spectrum:

- At $\omega \to 0$, one has $Q \to \infty$. Hence, large Q-balls possess soft modes. In this limit, the spectrum linearizes,

$$\gamma_n = k_n \omega \ , \qquad k_n \approx \frac{n\omega}{2} \ , \qquad n = 1, 3, 4, 5, ...$$

- The number of bound states of large Q-balls is proportional to its size[3].
- At intermediate frequencies the Q-balls do not support bound states.
- Close to the stability bound $\omega = \omega_c$ one vibrational spherically-symmetric mode reappears. For it

$$\gamma \sim \sqrt{\omega_c - \omega}$$

  This mode continues analytically into the instability region where it becomes the decay mode.

Perturbations of Q-balls in the flat potential

The structure of the spectrum with a non-zero orbital momentum is similar to that with $l = 0$:

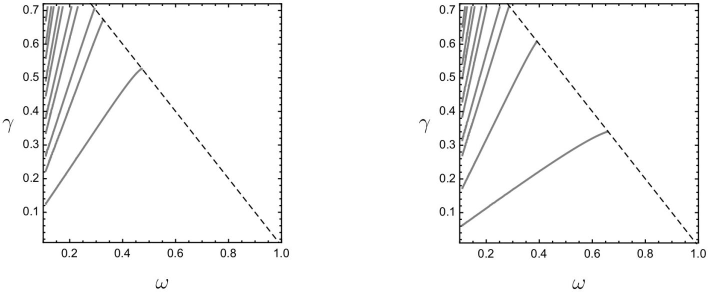


Figure: The discrete spectrum of linear perturbations of classically stable Q-balls in the flat potential, at $l = 1$ (the left panel) and $l = 2$ (the right panel).

Note the absence of vibrational modes near the cusp point $\omega = \omega_c$.

Polynomial potential

In order to allow Q-balls in a theory of one scalar field with a polynomial potential, it is necessary to include non-renormalizable self-interactions in the latter. Here we consider the simplest bounded below potential of the sixth degree,

Polynomial potential

$$V(|\phi|) = \left(\delta\left(|\phi|^2 - v^2\right)^2 + \omega_{\text{min}}^2\right)|\phi^2|\,, \qquad \delta > 0$$

The frequencies of Q-balls are confined in the region

$$\omega_{\text{min}} < \omega < m = \sqrt{\omega_{\text{min}}^2 + \delta\, v^4}$$

The thin-wall approximation is applicable near the lower limit. It is controlled by the small parameter

$$\epsilon = \omega - \omega_{\text{min}}$$

Q-balls in the thin-wall regime

In the thin-wall regime, the properties of a Q-ball are well captured by few quantities — the distance R to the wall and the magnitude f_0 of the field in the interior region.

In order to justify the description of a soliton in terms of a finite set of variables, a suitable thin-wall *ansatz* must be adopted.

To study perturbations on top of a Q-ball, it suffices to choose the simplest ansatz:

Thin-wall ansatz

$$f(r) = f_0 \theta \left(1 - \frac{r}{R}\right)$$

With this ansatz the energy and the charge of the Q-ball are

$$Q = \frac{8}{3}\pi R^3 \omega f_0^2 \ , \qquad E = 8\pi R^2 \sqrt{\delta} v^4 + \frac{4}{3}\pi R^3 \left(\omega^2 + \omega_{\text{min}}^2\right) f_0^2$$

Minimizing E while keeping Q fixed, one gets

$$f_0 = v + \mathcal{O}(\epsilon) \ , \qquad R = \frac{\sqrt{\delta} v^2}{2\omega_{\text{min}}} \frac{1}{\epsilon} + \mathcal{O}(1)$$

Perturbations in the thin-wall regime

The equations for perturbations are the same as before. The functions f and g are now given by

$$h(r) = -4\delta v^2 f^2(r) + 6\delta f^4(r) \, , \qquad g(r) = m^2 - 8\delta v^2 f^2(r) + 9\delta f^4(r)$$

The equations are disentangled in the exterior of the Q-ball, $r > R$. In the interior, $r < R$, one obtains separate equations for the rotated vector $\Xi = (\xi_1, \xi_2)^T$ such that

$$\Psi = U\Xi \, ,$$

where U diagonalizes the non-diagonal part of the linearized equations.

The resulting solutions are joined at $r = R$. This gives the spectrum of allowed values of γ.

Perturbations in the thin-wall regime

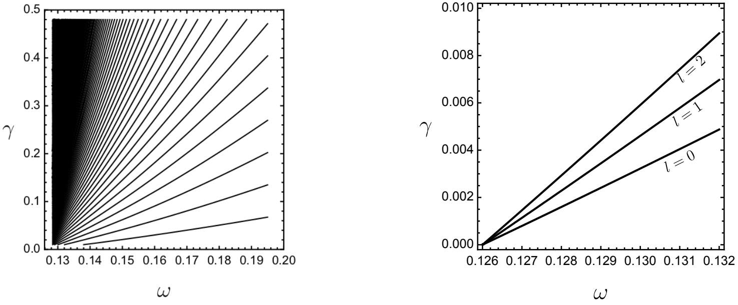


Figure: The spectrum of vibrational modes of stable Q-balls in the thin-wall approximation. The parameters of the potential are $\delta = 1.5$, $v = 0.9$, $\omega_{\text{min}} = 0.126$. All quantities are normalized to m. The left panel shows the full spectrum of the spherically-symmetric modes, $l = 0$. The right panel compares the modes of the 1st "energy level" and with different orbital momenta.

The features of the spectrum near the bound $\omega = \omega_{\text{min}}$ are the same as for the flat parabolic potential.

Conclusions

- The spectra of vibrations of the Q-balls in our examples have some properties in common. In fact, those properties are model-independent.
- Large Q-balls in the model with the flat potential possess soft modes with $\gamma \sim \omega \to 0$, well below the mass m of the free boson in vacuum.
- Q-balls with the near-critical charge have the vibrational mode related to the decay mode of Q-clouds.
- It is important to note that the near-critical regime of these (in general, relativistic) solitons can be analyzed by the means of the perturbation theory with respect to the relative frequency γ of an excitation.

Thank you!

Decay mode of Q-clouds in the flat potential

The decay mode is captured by the following spherically-symmetric ansatz,

$$\psi(\vec{x}, t) = \zeta(r) e^{\gamma t} \ , \qquad \gamma > 0$$

Define $\tilde{\gamma}$ as

$$\tilde{\gamma}^2 \equiv \gamma^2 \quad \text{for} \quad \omega < \omega_c \ , \qquad \tilde{\gamma}^2 \equiv -\gamma^2 \quad \text{for} \quad \omega \geq \omega_c$$

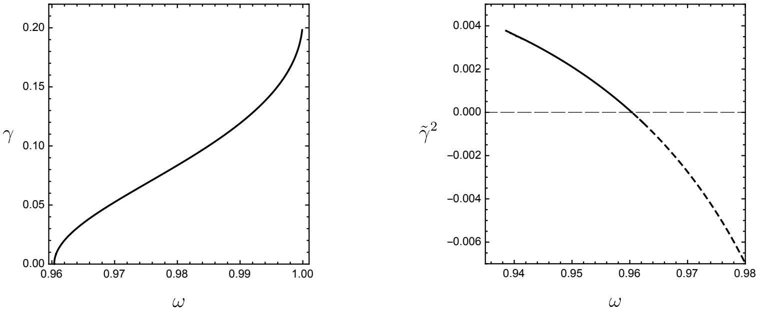


Figure: *Left panel:* the decay rate of unstable Q-balls in the flat parabolic potential. *Right panel:* the transition between the decay and the vibrational modes.

Perturbation theory near the cusp point

Whenever γ is small, one can make use of the perturbation theory with respect to γ. Then, the linear perturbations of a Q-ball take a simple form

$$\psi_1 \sim f + \gamma \frac{\partial f}{\partial \omega} + \mathcal{O}(\gamma^2) \,, \quad \psi_2 \sim -f + \gamma \frac{\partial f}{\partial \omega} + \mathcal{O}(\gamma^2)$$

Similarly, for the decay mode we have

$$\psi e^{-\gamma t} \sim if + \gamma \frac{\partial f}{\partial \omega} + \mathcal{O}(\gamma^2)$$

In this expression, the first term represents the Goldstone mode corresponding to the global $U(1)$-symmetry of the theory.