QCD splitting functions and cusp anomalous dimensions at four loops

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Based on work done in collaboration with:

- *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1805.09638`
- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1707.08315`
- Many more papers of MVV and friends . . .
  `2001 - ...`

Motivation

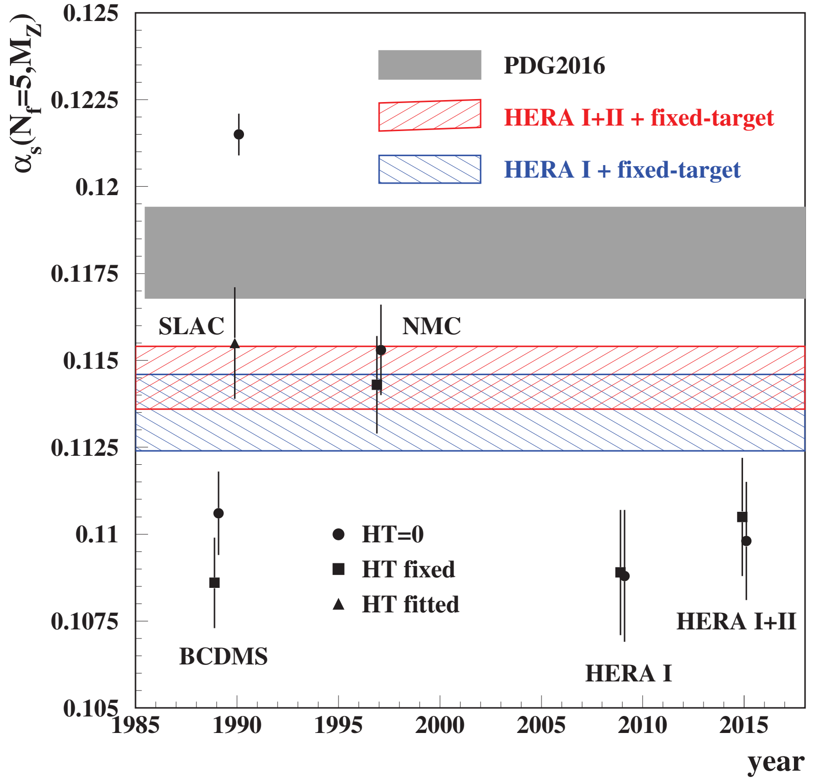
Strong coupling constant (2018)

BBG	$0.1134 \,^{+\,0.0019}_{-\,0.0021}$	valence analysis, NNLO	Blümlein, Böttcher, Guffanti '06
JR08	0.1128 ± 0.0010	dynamical approach	Jimenez-Delgado, Reya '08
	0.1162 ± 0.0006	including NLO jets	
ABKM09	0.1135 ± 0.0014	HQ: FFNS $n_f = 3$	Alekhin, Blümlein, Klein, S.M. '09
	0.1129 ± 0.0014	HQ: BSMN	
MSTW	0.1171 ± 0.0014		Martin, Stirling, Thorne, Watt '09
Thorne	0.1136	[DIS+DY, HT*] (2013)	Thorne '13
ABM11_J	$0.1134 \ldots 0.1149 \;\pm 0.0012$	Tevatron jets (NLO) incl.	Alekhin, Blümlein, S.M. '11
NN21	0.1173 ± 0.0007	(+ heavy nucl.)	NNPDF '11
ABM12	0.1133 ± 0.0011		Alekhin, Blümlein, S.M. '13
	0.1132 ± 0.0011	(without jets)	
CT10	0.1140	(without jets)	Gao et al. '13
CT14	$0.1150 \,^{+\,0.0060}_{-\,0.0040}$	$\Delta\chi^2 > 1$ (+ heavy nucl.)	Dulat et al. '15
JR14	0.1136 ± 0.0004	dynamical approach	Jimenez-Delgado, Reya '14
	0.1162 ± 0.0006	standard approach	
MMHT	0.1172 ± 0.0013	(+ heavy nucl.)	Martin, Motylinski, Harland-Lang, Thorne '15
ABMP16	0.1147 ± 0.0008		Alekhin, Blümlein, S.M., Placakyte '17

- Measurements at NNLO (last ~ 10 years) from DIS data
- Large spread of fitted values at NNLO: $\alpha_s(M_Z) = 0.1128 \ldots 0.1173$
- [yellow box] taken for 2016 PDG average: $\alpha_s(M_Z) = 0.1156 \pm 0.0021$

Theory considerations in α_s determinations

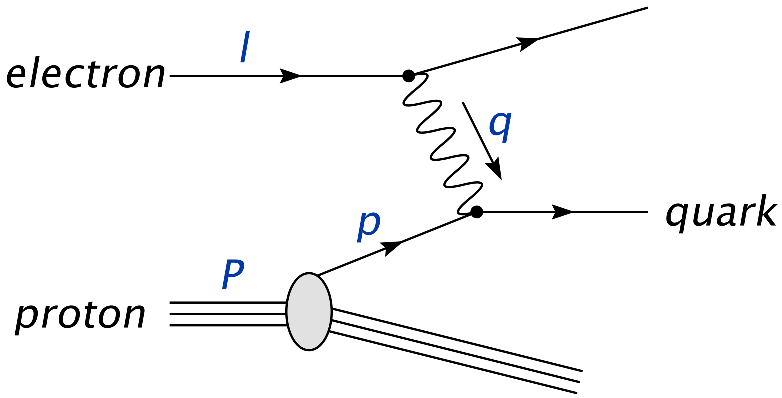
- Correlation of errors among different data DIS sets
- Target mass corrections (powers of nucleon mass M_N^2/Q^2)
- Higher twist $F_2^{\rm ht} = F_2 + ht^{(4)}(x)/Q^2 + \dots$
- Variants with no higher twist give larger α_s values Alekhin, Blümlein, S.M. '17



- Theoretical uncertainty of α_s at NNLO from DIS data $\gtrsim \mathcal{O}(1\dots2)\%$

Theoretical framework

Deep-inelastic scattering



Kinematic variables

- momentum transfer $Q^2 = -q^2$
- Bjorken variable $x = Q^2/(2p \cdot q)$

- Structure functions (up to order $\mathcal{O}(1/Q^2)$)

$$F_a\left(x, Q^2\right) = \sum_i \left[C_{a,i}\left(\alpha_s(\mu^2), \mu^2/Q^2\right) \otimes PDF\left(\mu^2\right)\right](x)$$

- Coefficient functions up to N^{3}LO

$$C_{a,i} = \alpha_s^n \left(c_{a,i}^{(0)} + \alpha_s c_{a,i}^{(1)} + \alpha_s^2 c_{a,i}^{(2)} + \alpha_s^3 c_{a,i}^{(3)} + \ldots\right)$$

- Evolution equations up to N^{3}LO
  - non-singlet ($2n_f - 1$ scalar) and singlet (2×2 matrix) equations

$$\frac{d}{d \ln \mu^2} PDF(x, \mu^2) \quad = \quad \left[P(\alpha_s(\mu^2)) \otimes PDF(\mu^2)\right](x)$$

  - splitting functions $P_{ij} = \alpha_s P_{ij}^{(0)} + \alpha_s^2 P_{ij}^{(1)} + \alpha_s^3 P_{ij}^{(2)} + \alpha_s^4 P_{ij}^{(3)} + \ldots$

Evolution equations

- Parton distribution functions $q_i(x,\mu^2)$, $\bar{q}_i(x,\mu^2)$ and $g(x,\mu^2)$ for quarks, antiquarks of flavour i and gluons
- Flavor non-singlet combinations

$$q^{\pm}_{\mathrm{ns},ik} \;=\; (q_i \pm \bar{q}_i) - (q_k \pm \bar{q}_k) \text{ and } q^{\mathrm{v}}_{\mathrm{ns}} \;=\; \sum_{i=1}^{n_f} (q_i - \bar{q}_i)$$

  - splitting functions $P^{\pm}_{\mathrm{ns}}$ and $P^{\mathrm{v}}_{\mathrm{ns}} = P^{-}_{\mathrm{ns}} + P^{\mathrm{s}}_{\mathrm{ns}}$
- Flavor singlet evolution

$$\frac{d}{d\ln\mu^2}\begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} = \begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} \quad \text{and} \quad q_{\mathrm{s}} \;=\; \sum_{i=1}^{n_f} (q_i + \bar{q}_i)$$

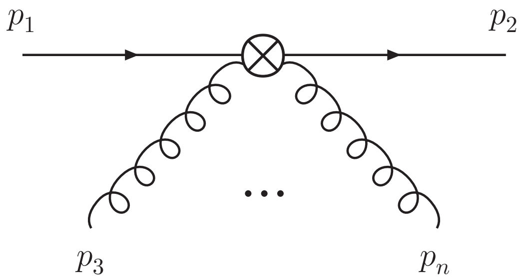
  - quark-quark splitting function $P_{\mathrm{qq}} = P^{+}_{\mathrm{ns}} + P_{\mathrm{ps}}$
- Mellin transformation relates to anomalous dimensions $\gamma_{\mathrm{ik}}(N)$ of twist-two operators

$$\gamma^{(n)}_{\mathrm{ik}}(N,\alpha_s) \;=\; -\int_0^1 dx\; x^{\,N-1}\, P^{(n)}_{\mathrm{ik}}(x,\alpha_s)$$

Non-singlet

Operator matrix elements

- Anomalous dimensions $\gamma(N)$ of leading twist non-singlet local operators
  - ultraviolet divergence of loop corrections to operator in (anti-)quark two-point function



  - expressible in harmonic sums up to weight 7

$$S_{\pm m_1,\dots,m_k}(N) = \sum_{i=1}^{N} \frac{(\pm 1)^i}{i^{m_1}} S_{m_2,\dots,m_k}(i)$$

  - $2 \cdot 3^{w-1}$ sums at weight w

- Reciprocity relation $\gamma(N) = \gamma_{\rm u}\left(N + \gamma(N) - \beta(a)\right)$ reduces number of 2^{w-1} sums at weight w for $\gamma_{\rm u}$
  - additional denominators with powers $1/(N+1)$ give $2^{w+1} - 1$ objects (255 at weight 7)

- Constraints at large-x/small-x ($N \to \infty / N \to 0$) give additional 46 conditions

Calculation

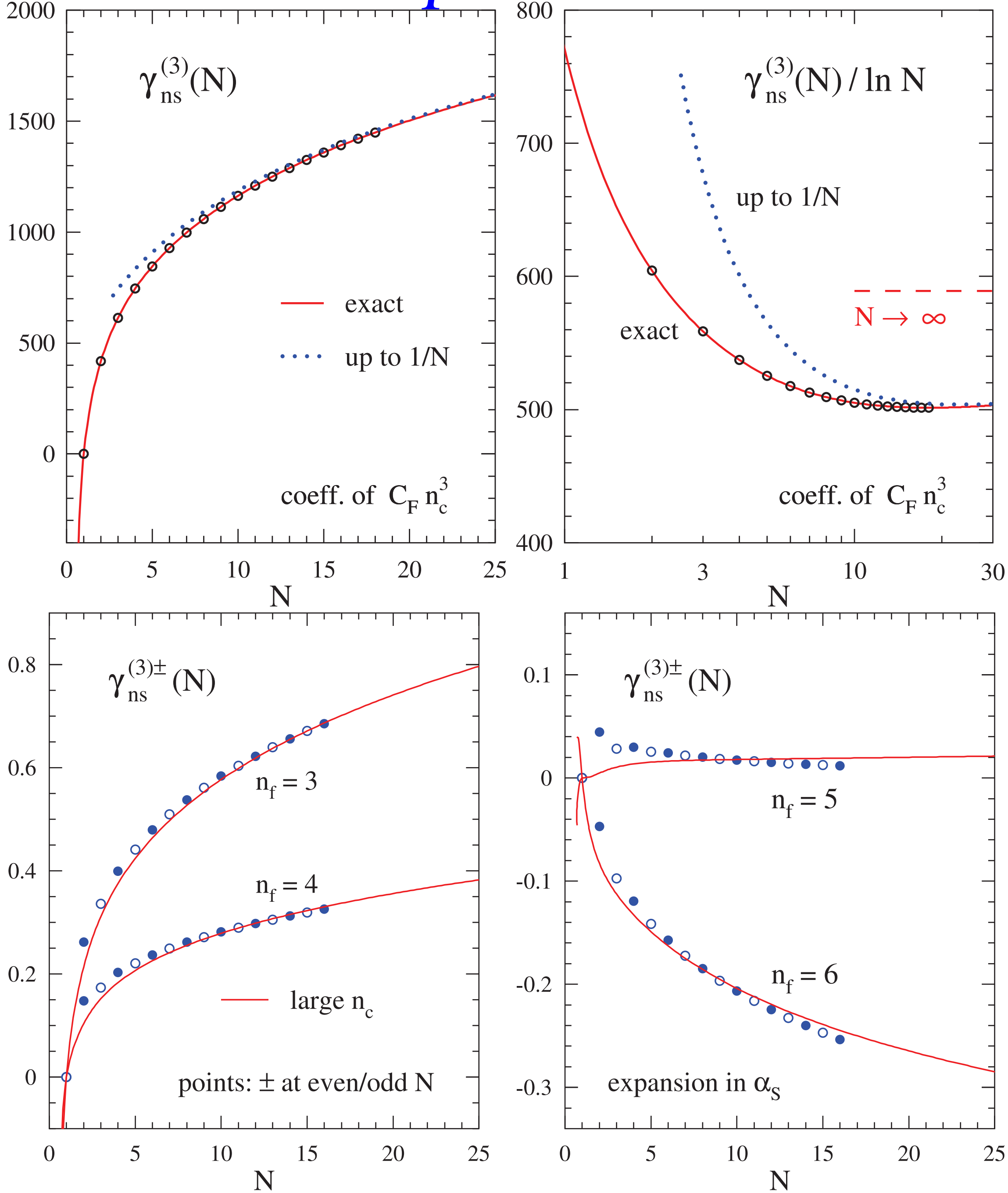
- Non-singlet operator of spin-N and twist two
  $O^{\rm ns}_{\{\mu_1,\dots,\mu_N\}} \;=\; \overline{\psi}\,\lambda^\alpha\,\gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}}\,\psi\;, \qquad \alpha = 3, 8, \dots, (n_f^2 - 1)$
- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira '91
- Parametric reduction of four-loop massless propagator diagrams with **Forcer** Ruijl, Ueda, Vermaseren '17
- Symbolic manipulations with **Form** Vermaseren '00; Kuipers, Ueda, Vermaseren, Vollinga '12 and multi-threaded version **TForm** Tentyukov, Vermaseren '07
- Diagrams of same topology and color factor combined to meta diagrams
  - 1 one-, 7 two-, 53 three- and 650 four-loop meta diagrams for $\gamma_{\rm ns}^{\pm}$
  - 1 three- and 29 four-loop meta diagrams for $\gamma_{\rm ns}^{\rm s}$

Upshot

- Computation of Mellin moments up to $N = 18$ for anomalous dimensions feasible
- Reconstruction of analytic all-N expressions in large-n_c limit from solution of Diophantine equations

Mellin moments at four loops

- Top:
  n_f^0 part of anomalous dimensions $\gamma_{\rm ns}^{(3)\pm}(N)$
  in large-n_c limit
  and large-N expansion

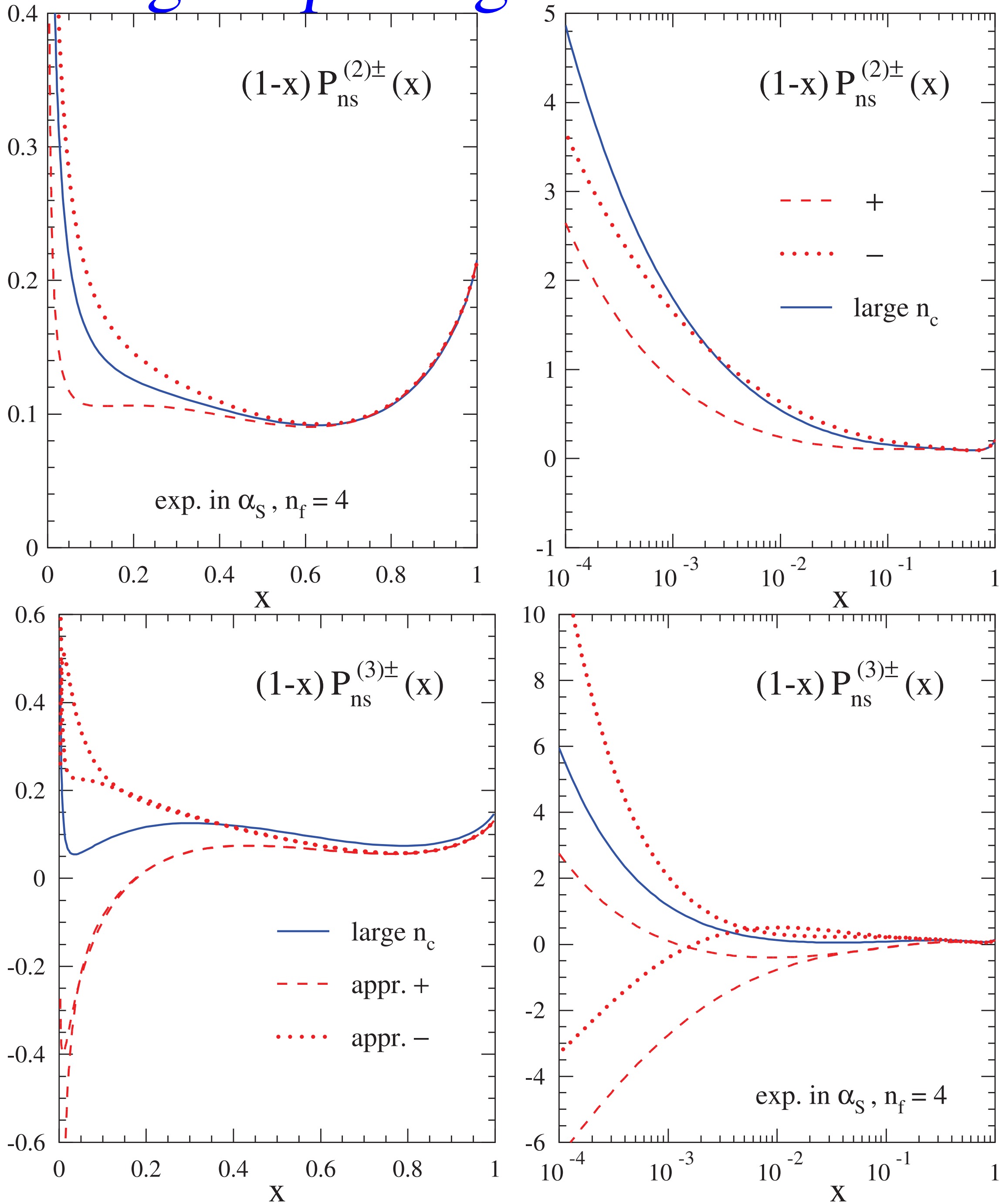


- Bottom: results for
  even-N ($\gamma_{\rm ns}^{(3)+}(N)$)
  and odd-N ($\gamma_{\rm ns}^{(3)-}(N)$)
  in large-n_c limit
  for $n_f = 3, \ldots 6$

Four-loop non-singlet splitting functions

- Top:
  three-loop $P_{\rm ns}^{(2)\pm}(x)$
  and large-n_c limit
  with $n_f = 4$

- Bottom:
  four-loop $P_{\rm ns}^{(3)\pm}(x)$
  and uncertainty bands
  beyond large-n_c limit
  with $n_f = 4$

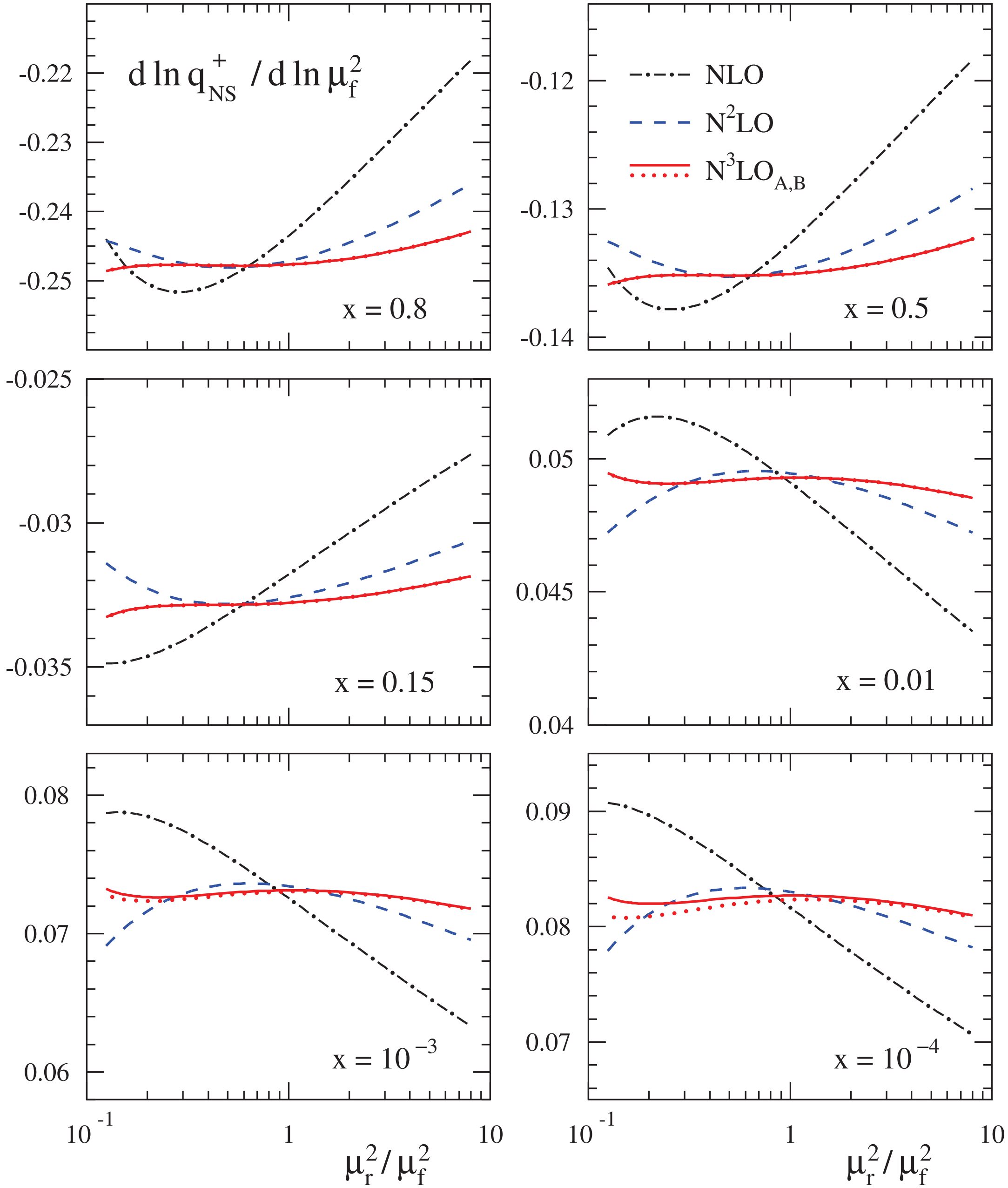


Scale stability of evolution

- Renormalization scale dependence of evolution kernel $d \ln q_{\rm ns}^{+} / d \ln \mu_f^2$
  - non-singlet shape

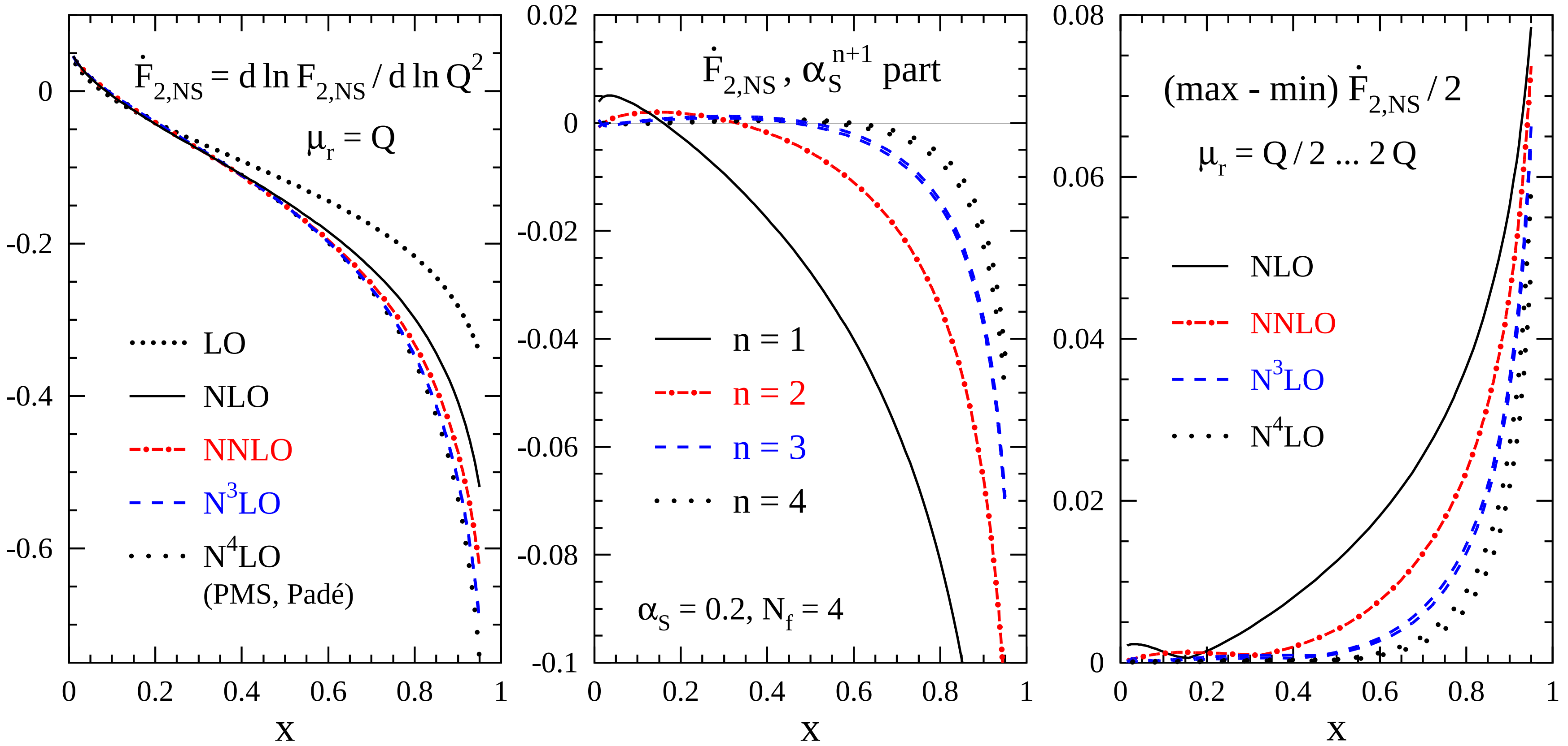
$$xq_{\rm ns}^{+}(x, \mu_0^2) \;=\; x^{0.5}(1-x)^3$$

- NLO, NNLO and N^{3}LO predictions
  - remaining uncertainty of four-loop splitting function $P_{\rm ns}^{(3)+}$ almost invisible



The structure function F_2 (non-singlet)

- Large-x convergence of perturbative series
  - approx. N^{3}LO structure functions S.M., Vermaseren, Vogt '05



- Potential for 'gold-plated' determination of α_s
  - theory uncertainty $\Delta_{\rm pert.}\alpha_s < 1\%$

Singlet

Color factors of $SU(n_c)$

- Quadratic Casimir factors $C_r \delta^{ab} \equiv \mathrm{Tr}\,(\, T_r^a T_r^b \,)$
  - fundamental representation $C_F = (n_c^2 - 1)/(2n_c)$;
    adjoint representation $C_A = n_c$
- Quartic Casimir invariants occur for the first time at four loops
  - $d_{xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd}$ for representations labels x, y with generators T_r^a

$$d_r^{abcd} = \frac{1}{6} \mathrm{Tr}\,(\, T_r^a T_r^b T_r^c T_r^d + \text{ five } bcd \text{ permutations} \,)$$

- SU(n_c) for fermions in fundamental representation

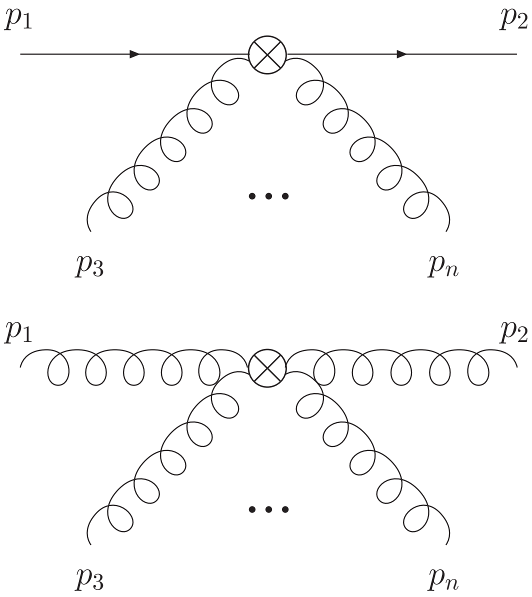
$$d_{AA}^{(4)}/n_A = \frac{1}{24} n_c^2 (n_c^2 + 36) \,,$$

$$d_{FA}^{(4)}/n_A = \frac{1}{48} n_c (n_c^2 + 6) \,,$$

$$d_{FF}^{(4)}/n_A = \frac{1}{96} (n_c^2 - 6 + 18 n_c^{-2})$$

  - trace-normalized with $T_F = \frac{1}{2}$
  - dimensions of representations $n_F = n_c$ and $n_A = (n_c^2 - 1)$

Operator matrix elements



- Singlet operators of spin-N and twist two

$$O^{\rm q}_{\{\mu_1,\dots,\mu_N\}} = \overline{\psi}\,\gamma_{\{\mu_1} D_{\mu_2} \dots D_{\mu_N\}}\,\psi\ ,$$
$$O^{\rm g}_{\{\mu_1,\dots,\mu_N\}} = F_{\nu\{\mu_1} D_{\mu_2} \cdots D_{\mu_{N-1}} F_{\mu_N\}}{}^{\nu}$$

- Quartic Casimir terms at four loops are effectively 'leading-order'
  - anomalous dimensions fulfil relation for $\mathcal{N}=1$ supersymmetry

$$\gamma^{(3)}_{\rm qq}(N) + \gamma^{(3)}_{\rm gq}(N) - \gamma^{(3)}_{\rm qg}(N) - \gamma^{(3)}_{\rm gg}(N) \stackrel{Q}{=} 0$$

- color-factor substitutions for $n_f = 1$ Majorana fermions (factor $2n_f$)

$$(2n_f)^2\,\frac{d_{FF}^{(4)}}{n_a} = 2n_f\,\frac{d_{FA}^{(4)}}{n_a} = 2n_f\,\frac{d_{FF}^{(4)}}{n_c} = \frac{d_{FA}^{(4)}}{n_c} = \frac{d_{AA}^{(4)}}{n_a}$$

- Eigenvalues of singlet splitting functions (conjectured to be) composed of reciprocity-respecting sums
  - quartic Casimir terms fulfil stronger, newly discovered condition

$$\gamma^{(0)}_{\rm qg}(N)\,\gamma^{(3)}_{\rm gq}(N) \stackrel{Q}{=} \gamma^{(0)}_{\rm gq}(N)\,\gamma^{(3)}_{\rm qg}(N)$$

Calculation and results

- Computation of Mellin moments up to $N = 16$ for anomalous dimensions feasible
- Reconstruction of analytic all-N expressions for ζ_5 terms from solution of Diophantine equations
  - example for $\gamma^{(3)}_{\rm gg}$ with $\eta = \frac{1}{N} - \frac{1}{N+1}$ and $\nu = \frac{1}{N-1} - \frac{1}{N+2}$

$$\gamma^{(3)}_{\rm gg}(N)\Big|_{\zeta_5\, d_{AA}^{(4)}/n_A} = \frac{64}{3}\Big(30\,(12\,\eta^2 - 4\,\nu^2 - S_1(4\,S_1 + 8\,\eta - 8\,\nu - 11) - 7\nu) + 188\,\eta - \frac{751}{3} - \frac{1}{6}\,N\,(N+1)\Big)$$

- Numerical approximaton x-space expressions
  - used for large-x limit

$$P^{(n-1)}_{ii}(x) = \frac{A_{n,\rm i}}{(1-x)_+} + B_{n,\rm i}\,\delta(1-x) + C_{n,\rm i}\,\ln(1-x) + D_{n,\rm i}$$

- Cusp anomalous dimensions $A_{\rm g}$
  - Casimir scaling up to three loops $A_{n,\rm g} = \frac{C_A}{C_F} A_{n,\rm q}$ for $n \le 3$

Quark and gluon cusp anomalous dimensions

- Large-n_c limit of quark cusp anomalous dimension (agrees with Henn, Lee, Smirnov, Smirnov, Steinhauser ‘16)

$$
\begin{aligned}
A_{4,\mathrm{q}}\big|_{\mathrm{large-n_c}} \;=\; & C_F n_c^3 \left(\frac{84278}{81} - \frac{88832}{81}\,\zeta_2 + \frac{20992}{27}\,\zeta_3 + 1804\zeta_4 - \frac{352}{3}\,\zeta_2\zeta_3 - 352\zeta_5 \right.\\
& \qquad \left. -32\zeta_3^2 - 876\,\zeta_6 \right)\\
& - C_F n_c^2\, n_f \left(\frac{39883}{81} - \frac{26692}{81}\,\zeta_2 + \frac{16252}{27}\,\zeta_3 + \frac{440}{3}\,\zeta_4 - \frac{256}{3}\,\zeta_2\zeta_3 - 224\,\zeta_5 \right)\\
& + C_F n_c\, n_f^2 \left(\frac{2119}{81} - \frac{608}{81}\,\zeta_2 + \frac{1280}{27}\,\zeta_3 - \frac{64}{3}\,\zeta_4 \right) - C_F n_f^3 \left(\frac{32}{81} - \frac{64}{27}\,\zeta_3 \right)
\end{aligned}
$$

- Result includes non-vanishing coefficients of quartic Casimir contributions $\dfrac{d_F^{abcd} d_A^{abcd}}{n_F}$ and $\dfrac{d_F^{abcd} d_F^{abcd}}{n_F}$

Generalized 'Casimir scaling'

quark	gluon	$A_{4,\mathrm{q}}$	$A_{4,\mathrm{g}}$
C_F^4	–	0	–
$C_F^3 C_A$	–	0	–
$C_F^2 C_A^2$	–	0	–
$C_F C_A^3$	C_A^4	610.25 ± 0.1	
$d_{FA}^{(4)}/N_F$	$d_{AA}^{(4)}/N_A$	-507.0 ± 2.0	-507.0 ± 5.0
$n_f\, C_F^3$	$n_f\, C_F^2 C_A$	-31.00554	
$n_f\, C_F^2 C_A$	$n_f\, C_F C_A^2$	38.75 ± 0.2	
$n_f\, C_F C_A^2$	$n_f C_A^3$	-440.65 ± 0.2	
$n_f\, d_{FF}^{(4)}/N_F$	$n_f\, d_{FA}^{(4)}/N_A$	-123.90 ± 0.2	-124.0 ± 0.6
$n_f^2\, C_F^2$	$n_f^2\, C_F C_A$	-21.31439	
$n_f^2\, C_F C_A$	$n_f^2\, C_A^2$	58.36737	
–	$n_f^2\, d_{FF}^{(4)}/N_A$	–	0.0 ± 0.1
$n_f^3\, C_F$	$n_f^3\, C_A$	2.454258	2.454258

- Numerical value for $n_f\, C_F^3$:
  - approximation: -31.00 ± 0.4 from S.M., Ruijl, Ueda, Vermaseren, Vogt '17
  - exact result: $-\,31.00554$ (rounded to seven digits) by Grozin '18

Numerical implications

- Numerical results (expansion in powers of $\alpha_s/(4\pi)$)

$$A_{4,\mathrm{q}} = 20702(2) - 5171.9(2)\,n_f + 195.5772\,n_f^2 + 3.272344\,n_f^3\,,$$
$$A_{4,\mathrm{g}} = 40880(30) - 11714(2)\,n_f + 440.0488\,n_f^2 + 7.362774\,n_f^3$$

  - Casimir scaling between $A_{4,\mathrm{g}}$ and $A_{4,\mathrm{q}}$ broken by almost 15% in n_f^0
  - non-leading large-n_c part of quartic-Casimir term (factors '36' and '6' in $A_{4,\mathrm{g}}$ and $A_{4,\mathrm{q}}$)

- Perturbative expansion very benign for quark

$$A_\mathrm{q}(\alpha_s, n_f=3) = 0.42441\,\alpha_s\,[1 + 0.72657\,\alpha_s + 0.73405\,\alpha_s^2 + 0.6647(2)\,\alpha_s^3 + \ldots]$$
$$A_\mathrm{q}(\alpha_s, n_f=4) = 0.42441\,\alpha_s\,[1 + 0.63815\,\alpha_s + 0.50998\,\alpha_s^2 + 0.3168(2)\,\alpha_s^3 + \ldots]$$
$$A_\mathrm{q}(\alpha_s, n_f=5) = 0.42441\,\alpha_s\,[1 + 0.54973\,\alpha_s + 0.28403\,\alpha_s^2 + 0.0133(3)\,\alpha_s^3 + \ldots]$$

and gluon

$$A_\mathrm{g}(\alpha_s, n_f=3) = 0.95493\,\alpha_s\,[1 + 0.72657\,\alpha_s + 0.73405\,\alpha_s^2 + 0.415(2)\,\alpha_s^3 + \ldots]$$
$$A_\mathrm{g}(\alpha_s, n_f=4) = 0.95493\,\alpha_s\,[1 + 0.63815\,\alpha_s + 0.50998\,\alpha_s^2 + 0.064(2)\,\alpha_s^3 + \ldots]$$
$$A_\mathrm{g}(\alpha_s, n_f=5) = 0.95493\,\alpha_s\,[1 + 0.54973\,\alpha_s + 0.28403\,\alpha_s^2 - 0.243(2)\,\alpha_s^3 + \ldots]$$

$N = 4$ *Super Yang-Mills theory*

$N = 4$ *Super Yang-Mills theory*

- The spectral problem anomalous dimension of local operators:
  $\mathcal{O}_n = \text{Tr}(\mathcal{W}_1 \mathcal{W}_2 \ldots \mathcal{W}_n)$ and $\mathcal{W}_i \in \{\mathcal{D}\Phi, \mathcal{D}\Psi, \mathcal{D}F\}$

$$\langle \mathcal{O}_a(x_1) \mathcal{O}_b(x_2) \rangle = \frac{\delta_{ab}}{(x_1 - x_2)^{2\Delta_a(\lambda)}}$$

- Spectrum of scaling dimensions: $\Delta(\lambda) = \Delta_0 + \gamma(\lambda)$
  - weak coupling expansion in limit: $n_c \to \infty$ and $\lambda = g^2 n_c$ fixed
- Dilatation operator corresponds to Heisenberg spin chain
  - solution with asymptotic Bethe ansatz
  - wrapping corrections when loop order $l \geq 2L$

Correspondence with QCD

- QCD results carry over to $N = 4$ SYM after substitution of color factors
  $C_A = C_F = n_c$ and so on
  - principle of “leading transcendentality”
    (keep only highest weight in ζ-function / harmonic sums)
  - at loops l-loops harmonic sums of weight $w = 2l - 1$

Universal anomalous dimension

- Universal anomalous dimension $\gamma_{\rm uni}$ in $N = 4$ SYM to three loops
  Kotikov, Lipatov, Onishchenko, Velizhanin '04
  - One-loop example: $\gamma_{\rm uni}^{(0)}(N) = 4 n_c S_1$ emerges from

$$\gamma_{\rm qq}^{(0)}(N) = C_F \left(-3 + 2 \frac{1}{N+1} - 2 \frac{1}{N} + 4 S_1 \right) \quad \text{or}$$

$$\gamma_{\rm gg}^{(0)}(N) = C_A \left(-\frac{11}{3} - \frac{4}{N-1} - \frac{4}{N+1} + \frac{4}{N+2} + \frac{4}{N} + 4 S_1 \right) + \frac{2}{3} n_f$$

- Starting at four loops wrapping corrections to complement asymptotic Bethe ansatz
  - control high-energy behaviour ($\sim 1/N^l$)
  - four-loop Bajnok, Janik, Lukowski '08, five-loop Lukowski, Rej, Velizhanin '09, six-loop [. . .], . . .
- $\gamma(N)^{\rm wrap,(4)} \simeq S_1(N)^2 \, f^{\rm wrap}(N)$

$$f^{\rm wrap}(N) \;=\; 5\zeta_5 - 2S_{-5} + 4S_{-2}\zeta_3 - 4S_{-2,-3} + 8S_{-2,-2,1} + 4S_{3,-2} - 4S_{4,1} + 2S_5$$

- Three-loop Wilson coefficient $c_{\rm ns}^{(3)}(N)$ S.M., Vermaseren, Vogt '05
  - $c_{\rm ns}^{(3)}(N) \simeq C_F \left(C_F - \frac{C_A}{2} \right)^2 \{ N(N+1) \, f^{\rm wrap}(N) \}$

Summary

- Determination of strong coupling α_s at 1% precision requires QCD radiative corrections to evolution equations at N^{3}LO
- Matrix elements of local operators of twist two
  - non-singlet anomalous dimensions $\gamma_{\rm ns}^{(3),\pm,{\rm v}}(N)$ (fixed Mellin moments and exact results for large-n_c) at N^{3}LO
  - quartic Casimir contributions to singlet anomalous dimension $\gamma_{ij}^{(3)}(N)$ (fixed Mellin moments and exact results for ζ_5 terms) at N^{3}LO
- Quark and gluon cusp anomalous dimensions
  - generalization of the lower-order ‘Casimir scaling’
- Structural similarities of QCD to $N = 4$ Super Yang-Mills theory