Conformally Coupled General Relativity

Andrej Arbuzov and Boris Latosh

BLTP, JINR, Dubna

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Outline

MOTIVATION

- Is it really necessary to modify GR?
- Nevertheless, how to quantize gravity?
- Conformal symmetry and its breaking?
- Time deparameterization?

Ogievetsky theorem

Any generator

$$L_k^{n_1,n_2,n_3,n_4} = -i x_0^{n_1} x_1^{n_2} x_2^{n_3} x_3^{n_4} \partial_k$$

of the infinite-dimensional diffeomorphism group Diff R^4 of general coordinate transformations

$$x'_\mu = x'_\mu(x_0, x_1, x_2, x_3)$$

can be represented as a linear combination of the commutators of the generators of the special linear and conformal groups:

$$\overbrace{R_{\mu\nu},\ L_{\mu,\nu},}^{SL(4,R)}\!\!\!\!\!\!\!\!\underbrace{\phantom{L_{\mu,\nu},}\ P_\mu,\ K_\mu,\ D}_{C(1,3)}$$

[V.I. Ogievetsky, Lett. Nuovo Cim. 1973]

NONLINEAR SYMMETRY REALIZATIONS

Let G be a d-dimensional Lie group and H its n-dimensional subgroup. Lie algebra of H is formed by generators V_l,; Lie algebra of G is formed by generators from algebra H and A_l, $(l = 1, \cdots, d-n)$. An arbitrary element g from a neighborhood of the identity of G can be cast as

$$g(\zeta, \varphi) = e^{\zeta_l A_l}\, e^{\varphi_l V_l}$$

Element $\exp[\varphi_l V_l]$ belongs to a neighborhood of the identity of H; element $\exp[\zeta_l A_l]$ belongs to the G/H coset. Restricted to the subgroup H we have a linear representation.

Nonlinear symmetry realizations are used to describe spontaneous symmetry breaking with yielding of Nambu–Goldstone bosons, e.g. the chiral symmetry breaking

[S.R. Coleman, J. Wess, B. Zumino, Phys. Rev. 1969]

Nonlinear realizations of affine and conformal symmetries

The Lorentz subgroup $SO(1,3)$ is chosen to be in linear realization.

Nonlinear realization of the affine group $\mathcal{A}(4)$ in the coset space over the Lorentz subgroup

$$\frac{\mathcal{A}(4)}{SO(1,3)} \sim \frac{P_m, L_{mn}, R_{mn}}{L_{mn}}$$

Nonlinear realization of the conformal group in the coset with the same stability subgroup

$$\frac{SO(2,4)}{SO(1,3)} \sim \frac{P_m, L_{mn}, K_n, D}{L_{mn}}$$

Simultaneous covariance under both nonlinear realizations was constructed (see review: [E.A. Ivanov, PEPAN 2016])

[A.B. Borisov, V.I. Ogievetsky, Theor. Math. Phys. 1975]

GR as a nonlinear realization

Einstein' gravity was obtained as a joint nonlinear realization of the affine and conformal symmetries with the Lorentz symmetry as the stability subgroup. The minimal invariant action coincides with the Einstein–Hilbert action

$$-\frac{1}{16\pi G}\int d^4x\sqrt{-g}R,$$

where the dimensionful Newton constant G appeared after re-scaling of the dimensionless Goldstone field h_{mn}.

Thus, graviton is both a gauge boson of the diffeomorphism group and a Goldstone mode due to spontaneous symmetry breaking. Dilaton also appears as a Goldstone related to scale invariance breaking.

[A.B. Borisov, V.I. Ogievetsky, Theor. Math. Phys. 1975]

CONFORMAL COUPLING

Deser (1970) and Dirac (1973) constructed conformal coupling of gravity to matter. The resulting action (Penrose; Chernikov and Tagirov) matches the standard Hilbert action up to a conformal transformation.

The scalar dilaton is extracted as a single degree of freedom from the full metric.

The standard metric g is governed by Einstein equation; it describes gravitational field self-dynamics. Metric $\tilde{g}$ is coupled to matter; it defines particles' motion

$$g_{\mu\nu} = \underbrace{e^{-2D}}_{\text{conformal scale factor}} \widetilde{g}_{\mu\nu}$$

THE ACTION

The conformal Hilbert–Einstein action with dilaton looks like a simple case of scalar-tensor Brans–Dicke model. Actually, it is not of this kind, since the transition to the Brans–Dicke case is singular (see Deser '70).

$$\int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{16\pi}(R-2\Lambda)\right]\to$$
$$\to\int d^4x\sqrt{-g}\left[\frac{\widetilde{M}_{Pl}^2}{16\pi}(\widetilde{R}-2\widetilde{\Lambda})+\frac{3\widetilde{M}_{Pl}^2}{8\pi}\widetilde{g}^{\mu\nu}\partial_\mu D\partial_\nu D\right]$$

The number of degrees of freedom is preserved, the dilaton is not a new "substance"

Vierbein basis

The unbroken Lorentz symmetry ⇒ projection on the Minkowskian tangent flat space by means of the tetrade (vierbein) formalism

$$g_{\mu\nu}dx^{\mu} \otimes dx^{\nu} = e^{-2D}\widetilde{g}_{\mu\nu}d\chi^{\mu} \otimes d\chi^{\nu} = e^{-2D}\eta_{(\mu)(\nu)}\omega_{(\mu)} \otimes \omega_{(\nu)}$$

where $\omega_{(\mu)}$ is the (conformal) vierbein basis subjected to the nonlinear symmetry realization

Interaction of gravity with matter including fermions is then straightforward but strictly conformal

[A.B. Arbuzov et al., Europhys. Lett. 2016]

(see also the talk by Tomislav Prokopec)

SPIN CONNECTION

The covariant derivative of a field Ψ reads

$$\begin{aligned} \nabla_{(\mu)}\Psi &= \omega_{(\mu)}^{\ \ \nu}\partial_\nu\Psi + \frac{i}{2}V_{(\mu),(\alpha)(\beta)}\,L^{\Psi}_{(\alpha)(\beta)}\Psi \\ V_{(\mu),(\alpha)(\beta)} &= \omega^L_{(\alpha)(\beta)}(\partial_{(\mu)}) + \omega^R_{(\alpha)(\mu)}(\partial_{(\beta)}) - \omega^R_{(\beta)(\mu)}(\partial_{(\alpha)}) \end{aligned}$$

with forms ω^L and ω^R are defined via tetrades as

$$\begin{aligned} \omega^R_{(\mu)(\nu)}(d) &= \frac{1}{2}\left(\omega_{(\mu)}^{\ \ \sigma}d\omega_{(\nu)\sigma} + \omega_{(\nu)}^{\ \ \sigma}d\omega_{(\mu)\sigma}\right) \\ \omega^L_{(\mu)(\nu)}(d) &= \frac{1}{2}\left(\omega_{(\mu)}^{\ \ \sigma}d\omega_{(\nu)\sigma} - \omega_{(\nu)}^{\ \ \sigma}d\omega_{(\mu)\sigma}\right) \end{aligned}$$

N.B. The forms will be treated as "primary" variables instead of the metric components

SYMMETRY CONDITION

The metric differential

$$dg_{\mu\nu} = d(\omega_{(\sigma)\mu}\omega_{(\sigma)\nu}) = \left(\omega_{(\rho)\mu}\omega_{(\sigma)\nu} + \omega_{(\sigma)\nu}\omega_{(\rho)\mu}\right)\left(\omega^{R}_{(\rho)(\sigma)}(d) + \omega^{L}_{(\rho)(\sigma)}(d)\right)$$

should by symmetric in $\mu \leftrightarrow \nu \Rightarrow$ the asymmetric form $\omega^{L}_{(\rho)(\sigma)}$ can't be a dynamical variable.

N.B. Construction of the covariant derivative in a nonlinear representation of $A(4)$ on the $A(4)/L$ coset is not unique, but combined representations of $A(4)$ and $C(1,3)$ resolves the problem [A.B. Borisov, V.I. Ogievetsky, Theor. Math. Phys. 1975]

Dirac-ADM foliation

There are serious arguments in favor of the Hamiltonian approach to gravity instead of the Lagrangian one. In particular, we need a global time for Cosmology.

So, we apply the standard Dirac–Arnowitt–Deser–Misner foliation

$$\widetilde{g}_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_a \\ N_b & \gamma_{ab} \end{pmatrix}$$

where N is the laps function, N_a is the shift vector, and γ_{ab} is the 3-metric

Time deparameterization

It was proposed to associate physical time with the zeroth dilaton mode in accord with the Einstein cosmological principle

$$\langle D\rangle(\chi^0) = \frac{1}{V_0}\int_{V_0} d^3\chi\,\sqrt{\gamma}\,D(\chi^0,\chi^1,\chi^2,\chi^3)$$

$$D(\chi^0,\chi^1,\chi^2,\chi^3) = \langle D\rangle(\chi^0) + \bar{D}(\chi^0,\chi^1,\chi^2,\chi^3), \qquad \langle\bar{D}\rangle = 0$$

The cosmological scale factor is identified

$$\langle D\rangle = -\ln a$$

Note that the conformal scale factor is universal: all objects are re-scaled strictly according to their conformal weights

[V.N. Pervushin et al. PLB 1997; GRG 1998]
see also [A.B. Arbuzov, A.E. Pavlov, arXiv:1710.01528]

ACTION DECOMPOSITION

Therefore, the CCGR action consists of three parts

$$
\begin{aligned}
S &= S_{\text{Universe}} + S_{\text{Gravitons}} + S_{\text{Potential}} \\
S_{\text{Universe}} &= -V_0 \int d\chi^0 N_0 \, \frac{3\widetilde{M}_P^2}{8\pi} \left(\frac{1}{N_0} \frac{\partial}{\partial \chi^0} \langle D \rangle \right)^2 \\
S_{\text{Gravitons}} &= \int d\chi^0 N_0 \int d^3\chi \sqrt{\gamma}\, \mathcal{N} \, \frac{\widetilde{M}_P^2}{16\pi} \widetilde{R} \\
S_{\text{Potential}} &= \int d\chi^0 N_0 \int d^3\chi \sqrt{\gamma}\, \mathcal{N} \, \frac{3\widetilde{M}_P^2}{8\pi} \gamma^{ab} \partial_a \bar{D} \partial_b \bar{D}
\end{aligned}
$$

where

$$
N(\chi^0, \chi^1, \chi^2, \chi^3) = N_0(\chi^0) \mathcal{N}(\chi^0, \chi^1, \chi^2, \chi^3)
$$

$$
\frac{1}{N_0} = \left\langle \frac{1}{N} \right\rangle , \qquad dt = N_0 d\chi^0
$$

TOWARDS QUANTIZATION OF CCGR

Principal point: the choice of physical variables to be quantized is dictated by symmetries. Revealing the conformal symmetry in GR helps here.

Instead of $g_{\mu\nu}$ we use tetrades as dynamical variables. Moreover instead of the Minkowskian background, we have to re-define gravitons in the cosmological background because of the dilaton presence

So, a single graviton state appears as a (nonlinear) plane wave propagating in the conformal space

Nonlinear plane wave (I)

The metric ansatz:

$$g = -d\chi^0 \otimes d\chi^0 + d\chi^3 \otimes d\chi^3 + e^{\Sigma}\left[e^{\sigma} d\chi^1 \otimes d\chi^1 + e^{-\sigma} d\chi^2 \otimes d\chi^2\right]$$

where $\sigma = \sigma(\chi^0, \chi^3)$ and $\Sigma = \Sigma(\chi^1, \chi^2)$

$$S_{\text{Gravitons}} \rightarrow \int d\chi^0\, d^3\chi\, \left\{ \frac{1}{2}\left[\left(\frac{\partial \sigma}{\partial \chi^0}\right)^2 - \left(\frac{\partial \sigma}{\partial \chi^3}\right)^2\right] - e^{-\Sigma}\left(e^{-\sigma}\frac{\partial^2 \Sigma}{\partial (\chi^1)^2} + e^{\sigma}\frac{\partial^2 \Sigma}{\partial (\chi^2)^2}\right)\right\}$$

NONLINEAR PLANE WAVE (II)

The ω^R form is simplified

$$\omega^R_{(\mu)(\nu)} = \omega_{(\mu)\sigma} d\omega_{(\nu)}^{\ \sigma} = \frac{1}{2} d\sigma \left(\delta_{(\mu)(1)} \delta_{(\nu)(1)} - \delta_{(\mu)(2)} \delta_{(\nu)(2)} \right)$$

Unlike metric variables, it admits the Taylor series expansion

$$\omega^R_{(a)(b)}(\partial_{(c)}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} ik_{(c)} \left[\epsilon^R_{(a)(b)}(k) g_k^+ e^{ik\cdot x} + \epsilon^R_{(a)(b)}(-k) g_k^- e^{-ik\cdot x} \right]$$

with the conditions

$$\epsilon^R_{(a)(a)}(k) = 0, \qquad k_{(a)} \epsilon^R_{(a)(b)}(k) = 0, \qquad k_\mu k^\mu = 0$$

THE CONFORMAL GRAVITON ACTION

The curvature scalar R is bilinear with respect to the external curvature, while the external curvature is linear with respect to ω^R

$$\Rightarrow$$

the graviton action is bilinear in ω^R, i.e. trivial

So, the introduced nonlinear gravitational waves do not interact with each other.

N.B. Their interaction with matter remains unchanged

[V.N. Pervushin et al., Gen. Rel. Grav. 2012; Phys. Atom. Nucl.2017]

THE CRUCIAL STEPS

- The joint nonlinear realization of $A(4)$ and conformal symmetries with the linear Lorentz subgroup
- The Dirac-ADM foliation with $\langle D \rangle = -\ln a$ identification
- The choice of the Maurer–Cartan forms ($\sim$ spin connections) as primary variables

Open problems and questions

- Cross checks of our approach
- Construction of the complete quantum model
- Renormalization behaviour and UV completeness
- The origin of the M_{Pl} scale
- Phenomenology: gravitational waves, Cosmology, conformal SM etc.