

# Lorentz Covariance in Higher-Spin Equations

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# Higher-Spin Theory

## Higher-Spin (HS) theory

- is an interacting theory of massless fields of all spins (including gravity)
- possesses an infinite-dimensional HS gauge symmetry (String Theory as broken phase?)
- lives on Anti-de Sitter background (has no sensible flat limit with HS symmetry unbroken)
- is dual to different boundary vectorial models (Klebanov–Polyakov conjecture)
- is formulated in the form of Vasiliev equations (action is unknown  $\Rightarrow$  direct check of AdS/CFT is unavailable)

# Structure of 4d Vasiliev equations

4d Vasiliev equations [Vasiliev, Phys.Lett. B285 (1992) 225]

$$\begin{aligned}d\mathcal{W} + \mathcal{W} * \mathcal{W} &= i\theta^A \theta_A + i\eta B * \gamma + i\bar{\eta} B * \bar{\gamma}, \\dB + [\mathcal{W}, B]_* &= 0.\end{aligned}$$

- $d = dx^{\underline{m}} \frac{\partial}{\partial x^{\underline{m}}}$  – space-time de Rham differential
- Master-fields

$$\mathcal{W}(Z, Y; K|x) = W_{\underline{m}}(Z, Y; K|x) dx^{\underline{m}} + S_A(Z, Y; K|x) \theta^A, \quad B(Z, Y; K|x).$$

- $Z^A$  and  $Y^A$  are twistor-like  $sp(4)$ -spinors contracting all indices of HS fields.
- Infinite number of HS interaction vertices are encoded into evolution in  $(Z, \theta)$ -directions.
- Physical fields –  $Z, \theta$ -independent components:

$$\omega(Y; K|x) = W_{\underline{m}}(Z=0, Y; K|x) dx^{\underline{m}}, \quad C(Y; K|x) = B(Z=0, Y; K|x).$$

- 1-form  $\omega$  contains all gauge-noninvariant d.o.f. (spin- $s$  gauge field and its first  $(s-1)$  derivatives for all  $s \geq 1$ )
- 0-form  $C$  contains all gauge-invariant d.o.f. (scalar and spinor fields, HS curvatures for  $s \geq 1$  and infinite towers of their descendants).

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- HS algebra is realized on twistor variables via star-product

$$(F * G)(Z, Y) = \frac{1}{(2\pi)^4} \int dU dV F(Z + U, Y + U) G(Z - V, Y + V) e^{iU_A V^A},$$

which is generalization of universal enveloping of Weyl algebra:

$$[Y_A, Y_B]_* = -[Z_A, Z_B]_* = 2i\epsilon_{AB}, \quad [Y_A, Z_B]_* = 0.$$

- Inner Klein operators  $\varkappa$  and  $\bar{\varkappa}$ :  $\varkappa = e^{iz_\alpha y^\alpha}$ ,  $\{\varkappa, y_\alpha\}_* = \{\varkappa, z_\alpha\}_* = 0$ ,  $\varkappa * \varkappa = 1$ .
- Outer Klein operators  $k$  and  $\bar{k}$ :  $\{k, y_\alpha\} = \{k, z_\alpha\} = \{k, \theta_\alpha\} = 0$ ,  $k^2 = 1$ .
- Central elements of the theory :

$$\delta^2(\theta) := \frac{1}{2} \theta^\alpha \theta_\alpha, \quad \gamma := 2k \varkappa \delta^2(\theta), \quad \delta^2(\bar{\theta}), \quad \bar{\gamma}.$$

# Lorentz covariance

- Noncovariant equations are handy due to  $d^2 \equiv 0$ . Though in AdS  $(D^L)^2 \neq 0$ , field redefinition to Lorentz-covariant formulation must exist in order to preserve equivalence principle.
- Lorentz-covariant derivative of a multispinor

$$D^L \phi_{\alpha(n), \dot{\alpha}(m)} = d\phi_{\alpha(n), \dot{\alpha}(m)} - n\omega_{\alpha}{}^{\beta} \phi_{\beta\alpha(n-1), \dot{\alpha}(m)} - m\bar{\omega}_{\dot{\alpha}}{}^{\dot{\beta}} \phi_{\alpha(n), \dot{\beta}\dot{\alpha}(m-1)},$$

$$(D^L)^2 \phi_{\alpha(n), \dot{\alpha}(m)} = nR_{\alpha}{}^{\beta} \phi_{\beta\alpha(n-1), \dot{\alpha}(m)} - m\bar{R}_{\dot{\alpha}}{}^{\dot{\beta}} \phi_{\alpha(n), \dot{\beta}\dot{\alpha}(m-1)},$$

$$R_{\alpha\beta} = d\omega_{\alpha\beta} - \omega_{\alpha}{}^{\gamma} \omega_{\gamma\beta}, \quad \bar{R}_{\dot{\alpha}\dot{\beta}} = d\bar{\omega}_{\dot{\alpha}\dot{\beta}} - \bar{\omega}_{\dot{\alpha}}{}^{\dot{\gamma}} \bar{\omega}_{\dot{\gamma}\dot{\beta}}, \quad D^L R_{\alpha\beta} = D^L \bar{R}_{\dot{\alpha}\dot{\beta}} = 0.$$

- In HS master-fields all spinor indices are contracted with  $Z^A$ ,  $Y^A$  or  $\theta^A$  and we have

$$D^L F(Z; Y; \theta) = \left( d + \omega^{AB} \left( Z_A \frac{\partial}{\partial Z^B} + Y_A \frac{\partial}{\partial Y^B} + \theta_A \frac{\partial}{\partial \theta^B} \right) \right) F(Z; Y; \theta).$$

- For  $\theta$ -independent functions there is a star-product realization of connection part

$$\omega^{AB} \left( Z_A \frac{\partial}{\partial Z^B} + Y_A \frac{\partial}{\partial Y^B} \right) = \omega^{AB} [L_{AB}, f]_* ,$$

$$L_{AB} = -\frac{i}{4} (Y_A Y_B - Z_A Z_B) .$$

# Lorentz symmetry in $\theta$ -sector

$\theta$ -sectors of Vasiliev equations

$$\begin{aligned}[\mathcal{S}_\alpha, \mathcal{S}_\beta]_* &= -2i\epsilon_{\alpha\beta} (1 + \eta B * \varkappa), \\ \{\mathcal{S}_\alpha, B * \varkappa\}_* &= 0.\end{aligned}$$

represents deformed oscillator algebra and respects  $sp(2)$  for any  $B$  [Vasiliev, Int.J.Mod.Phys. A6 (1991) 1115]:

$$\begin{aligned}M_{\alpha\beta} &= \frac{i}{8} \{\mathcal{S}_\alpha, \mathcal{S}_\beta\}_*, \\ [M_{\alpha\alpha}, M_{\beta\beta}]_* &= 2\epsilon_{\alpha\beta} M_{\alpha\beta}, \quad [M_{\alpha\alpha}, \mathcal{S}_\beta]_* = \epsilon_{\alpha\beta} \mathcal{S}_\alpha.\end{aligned}$$

It can be checked that the proper field redefinition, bringing HS equations to Lorentz-covariant form, is [Vasiliev hep-th/9910096].

$$W \rightarrow W + \omega^{AB} (L_{AB} + M_{AB}).$$

# Field redefinitions

Now we move the other way around. Assume there exists a field redefinition to Lorentz-covariant system

$$W \rightarrow W' = W + \omega^{AB} (\dots)$$

This should map

$$\begin{cases} dS_\alpha + [W, S_\alpha]_* = 0, \\ dB + [W, B]_* = 0, \end{cases} \longrightarrow \begin{cases} D^L S_\alpha + [W', S_\alpha]_* = 0, \\ D^L B + [W', B]_* = 0. \end{cases}$$

Consistency condition

$$(D^L)^2 S_\alpha = R^{\beta\gamma} [L_{\beta\gamma}, S_\alpha]_* - R_\alpha^\beta S_\beta$$

requires that

$$\left[ D^L W' + W' * W' + R^{\beta\gamma} \left( L_{\beta\gamma} - \frac{i}{4} S_\beta * S_\gamma \right), S_\alpha \right]_* = 0.$$

# Lorentz-covariant Vasiliev equations

Lorentz-covariant form of HS equations

$$D^L \mathcal{W} + \mathcal{W} * \mathcal{W} + R^{AB} \left( L_{AB} - \frac{i}{4} \frac{\partial}{\partial \theta^A} \mathcal{W} * \frac{\partial}{\partial \theta^B} \mathcal{W} \right) = i\theta^A \theta_A + i\eta B * \gamma + i\bar{\eta} B * \bar{\gamma},$$

$$D^L B + [\mathcal{W}, B]_* = 0,$$

$$R^{AB} := d\omega^{AB} - \omega^{AC} \omega_C{}^B.$$

Field redefinition relating this to initial noncovariant form

$$\mathcal{W}' = \mathcal{W} - \omega^{AB} \left( L_{AB} - \frac{i}{4} \frac{\partial}{\partial \theta^A} \mathcal{W} * \frac{\partial}{\partial \theta^B} \mathcal{W} \right).$$

Extra Stueckelberg symmetry gauging away Lorentz connection [[lazeolla, Sundell, JHEP 1112 (2011) 084]

$$\delta_\xi \omega_{AB} = \xi_{AB}, \quad \delta_\xi \mathcal{W}' = -\xi^{AB} \left( L_{AB} - \frac{i}{4} \frac{\partial}{\partial \theta^A} \mathcal{W} * \frac{\partial}{\partial \theta^B} \mathcal{W} \right), \quad \delta_\xi B = 0.$$

To fix  $\omega_{AB}$  we demand  $\mathcal{W}$  to contain no connection-type terms

$$\frac{\partial^2}{\partial y^\alpha \partial y^\beta} \mathcal{W}'|_{Z, Y=0} = \frac{\partial^2}{\partial \bar{y}^{\dot{\alpha}} \partial \bar{y}^{\dot{\beta}}} \mathcal{W}'|_{Z, Y=0} = 0.$$



# Extended HS equations

In [Vasiliev, Nucl.Phys. B916 (2017) 219] extension of  $4d$  HS equations was proposed, which includes higher forms and closed space-time functionals of HS fields (BH charges [Didenko, NM, Vasiliev, JHEP 1703 (2017) 164] and AdS/CFT functional presumably)

$$\mathbb{W} = \mathcal{W} + \mathcal{W}_3, \quad \mathbb{B} = B + B_2, \quad \mathcal{L}(x) := \mathcal{L}_2(x) + \mathcal{L}_4(x).$$

Extended system has the form

$$\begin{aligned} d\mathbb{W} + \mathbb{W} * \mathbb{W} &= i\theta^A \theta_A + i\eta \mathbb{B} * \gamma + i\bar{\eta} \mathbb{B} * \bar{\gamma} + ig\gamma * \bar{\gamma} + \mathcal{L}(x), \\ d\mathbb{B} + [\mathbb{W}, \mathbb{B}]_* &= 0, \\ d\mathcal{L}(x) &= 0, \end{aligned}$$

Higher forms do not influence sector of 1- and 0-forms and are expressed via them. The system possesses enhanced gauge symmetry

$$\begin{aligned} \delta\mathbb{W} &= d\epsilon + [\mathbb{W}, \epsilon]_* + i\eta\xi * \gamma + i\bar{\eta}\xi * \bar{\gamma} + \zeta, \\ \delta\mathbb{B} &= d\xi + \{\mathbb{W}, \xi\}_* + [\mathbb{B}, \epsilon]_* , \\ \delta\mathcal{L} &= d\zeta(x) . \end{aligned}$$

# Twisted sector and deformed oscillators

We expand twistor fields into components

$$\begin{aligned} \mathbb{W} \Big|_{dx=0} &= S_A \theta^A + 2t_\alpha \theta^\alpha \delta^2(\bar{\theta}) + 2\bar{t}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \delta^2(\theta), \\ \mathbb{B} \Big|_{dx=0} &= B + 2b\delta^2(\theta) + 2\bar{b}\delta^2(\bar{\theta}) + b_{\alpha\dot{\alpha}} \theta^\alpha \bar{\theta}^{\dot{\alpha}}. \end{aligned}$$

Then HS equations take the form

$$\begin{aligned} [S_\alpha, S_\beta]_* &= -2i\epsilon_{\alpha\beta} (1 + \eta B * \varkappa), \quad [S_\alpha, \bar{S}_{\dot{\alpha}}]_* = 0, \\ [S_\alpha, \bar{b}]_* + [t_\alpha, B]_* &= 0, \quad [S_\alpha, B]_* = 0, \quad \frac{1}{2} [S_\alpha, b^{\alpha\dot{\alpha}}]_* + [\bar{t}_{\dot{\alpha}}, B]_* = 0, \\ [t_\alpha, S^\alpha]_* + [\bar{t}_{\dot{\alpha}}, \bar{S}^{\dot{\alpha}}]_* &= -2i (\eta \bar{b} * \varkappa + \bar{\eta} b * \bar{\varkappa} + g \varkappa * \bar{\varkappa}). \end{aligned}$$

It turns out that this system admits realization of Lorentz symmetry as gauge transformation. For  $\mathbb{B} = 0$  case gauge symmetries are

$$\begin{aligned} \delta_\Lambda S_\alpha &= [S_\alpha, \epsilon]_* , \\ \delta_\Lambda t_\alpha &= [t_\alpha, \epsilon]_* + [\phi, S_\alpha] + [\psi_\alpha^{\dot{\beta}}, \bar{S}_{\dot{\beta}}] , \\ \delta_\Lambda \bar{t}_{\dot{\alpha}} &= [\bar{t}_{\dot{\alpha}}, \epsilon]_* + [\bar{\phi}, \bar{S}_{\dot{\alpha}}] - [\bar{\psi}^{\beta\dot{\alpha}}, S_\beta] . \end{aligned}$$

Then choosing

$$\epsilon = -\frac{i}{4} \Lambda^{\alpha\beta} S_\alpha * S_\beta, \quad \phi = \frac{i}{4} \Lambda^{\alpha\beta} \{S_\alpha * t_\beta\}, \quad \psi_{\alpha\dot{\alpha}} = -\frac{i}{4} \Lambda_\alpha{}^\beta \{S_\beta, \bar{t}_{\dot{\alpha}}\}_*$$

we get

$$\delta_\Lambda S_\alpha = \Lambda_\alpha{}^\beta S_\beta, \quad \delta_\Lambda t_\alpha = \Lambda_\alpha{}^\beta t_\beta.$$

# Generalized deformed oscillator algebra

Oscillator (Weyl) algebra:

$$[y_\alpha, y_\beta] = -2i\epsilon_{\alpha\beta}.$$

Deformed oscillator algebra:

$$[y_\alpha, y_\beta] = -2i\epsilon_{\alpha\beta} (1 + \nu K), \quad \{y_\alpha, K\} = 0, \quad KK = 1.$$

For any value of central element  $\nu$   $y$ -bilinears form  $sp(2)$

$$M_{\alpha\beta} = \{y_\alpha, y_\beta\}, \quad [M_{\alpha\alpha}, M_{\beta\beta}] \sim \epsilon_{\alpha\beta} M_{\alpha\beta}.$$

Adding another copy  $\bar{y}$  generate full Lorentz algebra  $sp(2|\mathbb{C})$ . Extended equations provide a nontrivial generalization of deformed oscillators that still respects  $sp(2|\mathbb{C})$ . For  $B = \text{const}$ ,  $b = 0$  it is

$$\begin{aligned} [S_\alpha, S_\beta] &= -2i\epsilon_{\alpha\beta} (1 + \nu K), & [\bar{S}_{\dot{\alpha}}, \bar{S}_{\dot{\beta}}] &= -2i\epsilon_{\dot{\alpha}\dot{\beta}} (1 + \nu \bar{K}), \\ \{S_\alpha, K\} &= 0, & \{\bar{S}_{\dot{\alpha}}, \bar{K}\} &= 0, \\ \{t_\alpha, K\} &= 0, & \{\bar{t}_{\dot{\alpha}}, \bar{K}\} &= 0, \\ [S_\alpha, \bar{S}_{\dot{\alpha}}] &= 0, & [t_\alpha, S^\alpha] + [\bar{t}_{\dot{\alpha}}, \bar{S}^{\dot{\alpha}}] &= 2igK\bar{K}. \end{aligned}$$

# Lorentz-covariant ext-HS system

Once again, assuming the existence of field redefinition and analyzing consistency condition in twisted sector, we get Lorentz-covariant form of extended HS equations

$$\begin{aligned}
 D^L \mathbb{W} + \mathbb{W} * \mathbb{W} + R^{AB} \left( L_{AB} - \frac{i}{4} \frac{\partial}{\partial \theta^A} \mathbb{W} * \frac{\partial}{\partial \theta^B} \mathbb{W} \right) &= i\theta^A \theta_A + i\eta \mathbb{B} * \gamma + i\bar{\eta} \mathbb{B} * \bar{\gamma} + i\mathbf{g}\gamma * \bar{\gamma} + \mathcal{L} - \\
 - \left( \frac{\eta}{4} R^{\alpha\beta} \frac{\partial}{\partial \theta^\alpha} \mathbb{B} * \frac{\partial}{\partial \theta^\beta} \gamma - \frac{i\eta}{32} R^{\alpha\alpha} R^{\beta\beta} \frac{\partial^2}{\partial \theta^\alpha \partial \theta^\beta} \mathbb{B} * \frac{\partial^2}{\partial \theta^\alpha \partial \theta^\beta} \gamma + \text{c.c.} \right) \\
 D^L \mathbb{B} + [\mathbb{W}, \mathbb{B}]_* - \frac{i}{4} R^{AB} \left\{ \frac{\partial}{\partial \theta^A} \mathbb{B}, \frac{\partial}{\partial \theta^B} \mathbb{W} \right\}_* &= 0.
 \end{aligned}$$

Stueckelberg symmetry switching between covariant and noncovariant systems

$$\begin{aligned}
 \delta_\xi \omega_{AB} &= \xi_{AB}, \quad \delta_\xi \mathbb{B} = \frac{i}{4} \xi^{AB} \left\{ \frac{\partial}{\partial \theta^A} \mathbb{B}, \frac{\partial}{\partial \theta^B} \mathbb{W} \right\}_* \\
 \delta_\xi \mathbb{W} &= -\xi^{\alpha\beta} \left( L_{\alpha\beta} - \frac{i}{4} \frac{\partial}{\partial \theta^\alpha} \mathbb{W} * \frac{\partial}{\partial \theta^\beta} \mathbb{W} \right) - \frac{\eta}{4} \xi^{\alpha\beta} \frac{\partial}{\partial \theta^\alpha} \mathbb{B} * \frac{\partial}{\partial \theta^\beta} \gamma + \\
 + \frac{i\eta}{32} \left( \xi^{\alpha\alpha} R^{\beta\beta} + \xi^{\beta\beta} R^{\alpha\alpha} \right) \frac{\partial^2}{\partial \theta^\alpha \partial \theta^\beta} \mathbb{B} * \frac{\partial^2}{\partial \theta^\alpha \partial \theta^\beta} \gamma + \text{c.c.}
 \end{aligned}$$

# Lorentz symmetry and central elements

- Consider noncovariant non-extended equation

$$d\mathcal{W} + \mathcal{W} * \mathcal{W} = i\theta^A \theta_A + i\eta B * \gamma + i\bar{\eta} B * \bar{\gamma}.$$

Rescaling properly  $\theta$  and master-fields one can set central element to be  $i\nu\theta^A\theta_A$ .  
Covariantized equation becomes

$$D^L \mathcal{W} + \mathcal{W} * \mathcal{W} + R^{AB} \left( L_{AB} - \frac{i}{4\nu} \frac{\partial}{\partial \theta^A} \mathcal{W} * \frac{\partial}{\partial \theta^B} \mathcal{W} \right) = i\theta^A \theta_A + i\eta B * \gamma + i\bar{\eta} B * \bar{\gamma}.$$

So limit  $\nu \rightarrow 0$  is forbidden by requirement of Lorentz symmetry.

- Consider Lorentz-covariant extended equation with  $ig\gamma * \bar{\gamma}$  replaced by general 4-form central element  $c$ .

$$D^L \mathbb{W} + \mathbb{W} * \mathbb{W} + R^{AB} \left( L_{AB} - \frac{i}{4} \frac{\partial}{\partial \theta^A} \mathbb{W} * \frac{\partial}{\partial \theta^B} \mathbb{W} \right) = i\theta^A \theta_A + i\eta \mathbb{B} * \gamma + i\bar{\eta} \mathbb{B} * \bar{\gamma} + c + \mathcal{L} -$$

$$- \left( \frac{\eta}{4} R^{\alpha\beta} \frac{\partial}{\partial \theta^\alpha} \mathbb{B} * \frac{\partial}{\partial \theta^\beta} \gamma - \frac{i\eta}{32} R^{\alpha\alpha} R^{\beta\beta} \frac{\partial^2}{\partial \theta^\alpha \partial \theta^\beta} \mathbb{B} * \frac{\partial^2}{\partial \theta^\alpha \partial \theta^\beta} \gamma + c.c. \right)$$

Consistency in particular requires

$$R^{AB} \left\{ \frac{\partial}{\partial \theta^A} c, \frac{\partial}{\partial \theta^B} \mathbb{W} \right\} = 0$$

which holds for  $c = ig\gamma * \bar{\gamma}$  but not for other central elements (like  $\delta^4(\theta)$ ).

# Conclusions

- We showed that extended HS equations admit local Lorentz symmetry.
- We find that requirement of Lorentz symmetry restricts the form of central elements entering HS equations.
- We find that extended HS equations provide a nontrivial generalization of deformed oscillator algebra.
- We find that operator of HS perturbation theory are drastically simplified in Lorentz-covariant formulation (stayed outside the talk, but it's really true!)