

Ricci-flat metrics and Killing-Yano tensors

Dmitri Bykov

*Ludwig-Maximilians-Universität (München) &
Steklov Mathematical Institute (Moscow)*

Quarks 2018, Valday, 30.05.2018

Part I. General facts.

This talk will be about Calabi-Yau threefolds \mathcal{M}

- Complex manifolds of complex dimension three: $\dim_{\mathbb{C}} \mathcal{M} = 3$
- Zero first Chern class: $c_1(\mathcal{M}) = c_1(K) = 0$
(K is the canonical bundle = bundle of 3-forms $\Omega \propto f(z) dz_1 \wedge dz_2 \wedge dz_3$), i.e. there exists a non-vanishing holomorphic 3-form Ω)
- Such manifolds are used for supersymmetric compactifications in supergravity ($\mathbb{R}^{3,1} \times \mathcal{M}$), and serve as backgrounds for brane constructions ($AdS_5 \times Y^5$)

Non-compact Calabi-Yau manifolds

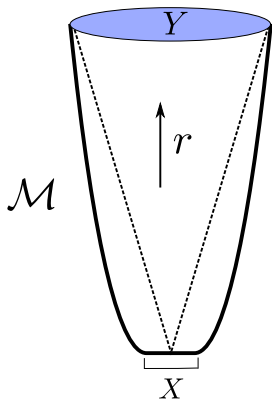
It is easy to show that compact Calabi-Yau's do not admit Killing vectors (apart from trivial cases), therefore explicit metrics are difficult to construct.

This talk will be about non-compact Calabi-Yau's, which **do** have symmetries. In this case the geometry of such manifolds may often be studied explicitly. These non-compact Calabi-Yau's may be thought of as describing singularities of compact Calabi-Yau's.

Let X be a positively curved complex surface, $c_1(X) > 0$. Here one should recall that $c_1(X) = \left[\frac{i}{2\pi} R_{m\bar{n}} dz^m \wedge d\bar{z}^{\bar{n}} \right] \in H^2(X, \mathbb{R})$. We will be studying the case

$\mathcal{M} =$ Total space of the canonical bundle of $X =$ “Cone over X ”.

Non-compact Calabi-Yau manifolds



The corresponding singularity is pointlike and may be then resolved by gluing in a copy of X .

This is just like the prototypical $\mathbb{C}^2/\mathbb{Z}_2$ -singularity (“ A_1 -singularity”) given by equation $xy = z^2$ may be resolved by gluing in a copy of \mathbb{CP}^1 at the origin. The metric on the resolved space is then the Eguchi-Hanson metric. (However, this corresponds to \mathcal{M} of complex dimension 2.)

First example. Calabi's ansatz.

If X admits a Kähler-Einstein metric, the metric on \mathcal{M} may be found by means of an ansatz [Calabi \('79\)](#)

$$\mathcal{K} = \mathcal{K}(|u|^2 e^K),$$

where \mathcal{K} and K are the Kähler potentials of \mathcal{M} and X respectively. The Ricci-flatness equation becomes in this case an ODE for the function $\mathcal{K}(x)$.

For example, for $X = \mathbb{C}P^2$ one obtains in this way the (generalized) Eguchi-Hanson metric. [Eguchi, Hanson \('78\)](#)

These metrics are asymptotically-conical, i.e. they have the form

$$ds^2 = dr^2 + r^2 (\widetilde{ds^2})_Y \quad \text{at} \quad r \rightarrow \infty,$$

where $(\widetilde{ds^2})_Y$ is a Sasaki-Einstein metric on a 5D real manifold Y .

Calabi's ansatz.

An important characteristic of a Kähler metric on \mathcal{M} is the cohomology class $[\omega] \in H^2(\mathcal{M}, \mathbb{R})$ of the Kähler form. Since \mathcal{M} is a total space of a line bundle, its cohomology is the same as that of the underlying surface X . Therefore, for instance for $X = \mathbb{C}P^2$ we have $H^2(\mathcal{M}, \mathbb{R}) = \mathbb{R}$, but for $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ we have $H^2(\mathcal{M}, \mathbb{R}) = \mathbb{R}^2$.

Calabi's ansatz gives a metric with a very particular and fixed $[\omega] \in H^2(\mathcal{M}, \mathbb{R})$. It turns out that $[\omega] \in H_c^2(\mathcal{M}, \mathbb{R}) \subset H^2(\mathcal{M}, \mathbb{R})$, where H_c^2 is the compactly supported cohomology. By Poincaré duality, the group $H_c^2(\mathcal{M}, \mathbb{R}) \simeq H_4(\mathcal{M}, \mathbb{R}) = H_4(X, \mathbb{R}) = \mathbb{R}$ is one-dimensional.

The Calabi-Yau theorem.

The Calabi-Yau theorem [Calabi \('57\)](#), [Yau \('79\)](#) states, however, that, at least for compact \mathcal{M} , there is a unique Ricci-flat metric in **every** Kähler class $[\omega] \in H^2(\mathcal{M}, \mathbb{R})$.

For the case of interest \mathcal{M} is not compact, but asymptotically-conical, and in this case there exists a proposal for a CY theorem due to [van Coevering \('2008\)](#), [Goto \(2012\)](#). Moreover, one has the decay estimates

$$\begin{aligned} |g - g_0|_{g_0} &= O\left(\frac{1}{r^6}\right) && \text{for } [\omega] \in H_c^2(\mathcal{M}, \mathbb{R}) \\ |g - g_0|_{g_0} &= O\left(\frac{1}{r^2}\right) && \text{for } [\omega] \in H^2(\mathcal{M}, \mathbb{R}) \setminus H_c^2(\mathcal{M}, \mathbb{R}), \end{aligned}$$

where g_0 is the conical metric. Such estimates were introduced for the case of ALE-manifolds in [Joyce \('99\)](#).

Example. $X = \mathbb{C}P^1 \times \mathbb{C}P^1$.

The theory just described can be tested explicitly at the example of $X = \mathbb{C}P^1 \times \mathbb{C}P^1$. The ansatz for the Kähler potential on the cone over X is a generalized ansatz of Calabi constructed by [Candelas, de la Ossa \('90\)](#), [Pando Zayas, Tseytlin \('2001\)](#):

$$\mathcal{K} = a \log(1 + |w^2|) + \mathcal{K}_0 \left(|u^2|(1 + |w^2|)(1 + |x^2|) \right).$$

The resulting metric, indeed, has two parameters that define the cohomology class of the Kähler form $[\omega] \in H^2(\mathcal{M}, \mathbb{R}) = \mathbb{R}^2$. These correspond to the sizes of the two spheres. The relevant Sasakian manifold Y at $r \rightarrow \infty$ is the conifold $T^{1,1} = \frac{SU(2) \times SU(2)}{U(1)}$, and the decay at infinity agrees with the predicted one.

Part II. The del Pezzo surface of rank one.

The del Pezzo surface

We will be interested in the next-to-simplest example:

$X =$ del Pezzo surface of rank one

(= Hirzebruch surface of rank one) = the blow-up of $\mathbb{C}P^2$ at one point.

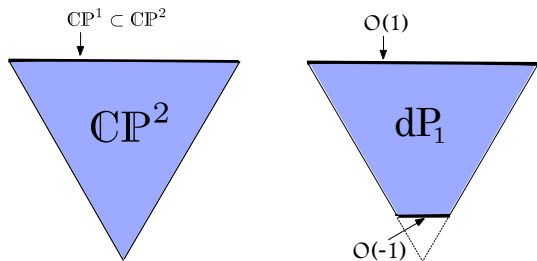


Pasquale del Pezzo (1859-1936),
Rector of the University of Naples,
Mayor of Naples, Senator

Del Pezzo surfaces ('1887) are natural generalizations to higher complex dimensions of positively curved Riemann surfaces (the sphere $S^2 = \mathbb{C}P^1$) and thus are very special.

Metrics on the del Pezzo surface

A blow-up means that we replace one point in $\mathbb{C}P^2$ by a sphere $\mathbb{C}P^1$. This $\mathbb{C}P^1$ 'remembers the direction', at which we approach the point. A 'good' metric on the new manifold should have two parameters, which describe the original size of the $\mathbb{C}P^2$ and the size of the glued in sphere $\mathbb{C}P^1$. The del Pezzo surface is a toric manifold, and the best way to think of it is via its moment polygon.



Metrics on the cone and toric geometry

A theorem of [Tian, Yau \('87\)](#) says that there does **not** exist a Kähler-Einstein metric on \mathbf{dP}_1 . How do we then construct a metric on the cone \mathcal{M} over \mathbf{dP}_1 ? The only hope is to use its symmetries, which are those symmetries of $\mathbb{C}\mathbb{P}^2$ that remain after the blow-up.

The relevant isometry group is $U(1) \times U(2)$, however for the moment let us focus on the toric $U(1)^3$ subgroup. Generally, the Kähler potential has the form

$$\mathcal{K} = \mathcal{K} \left(\underbrace{|z_1|^2}_{=e^{t_1}}, \underbrace{|z_2|^2}_{=e^{t_2}}, \underbrace{|z_3|^2}_{=e^{t_3}} \right).$$

Metrics on the cone and toric geometry

It is customary to introduce the symplectic potential \mathcal{G} – the Legendre transform of the Kähler potential w.r.t. t_i :

$$\mathcal{G}(\mu_1, \mu_2, \mu_3) = \sum_{j=1}^3 \mu_j t_j - \mathcal{K}$$

Here $\mu_i = \frac{\partial \mathcal{K}}{\partial t_i}$ are the moment maps for the $U(1)^3$ symmetries of the problem. The metric on \mathcal{M} has the form

$$ds^2 = \frac{1}{4} \mathcal{G}_{ij} d\mu^i d\mu^j + (\mathcal{G}^{-1})^{ij} d\phi_i d\phi_j.$$

The Riemann tensor with all lower indices looks as follows:

$$R_{\bar{m}j k \bar{n}} = - \sum_{s,t} \mathcal{G}_{ns}^{-1} \frac{\partial^2 \mathcal{G}_{jk}^{-1}}{\partial \mu_s \partial \mu_t} \mathcal{G}_{tm}^{-1}.$$

Metrics on the cone and toric geometry

The domain, on which \mathcal{G} is defined, is the moment polytope. The potential \mathcal{G} has singularities at the boundaries of the polytope. For instance, for flat space \mathbb{C}^3 the polytope is the octant, and \mathcal{G} has the form

$$\mathcal{G}_{\text{flat}} = \sum_{k=1}^3 \mu_k (\log \mu_k - 1).$$

In general, at a boundary $L = 0$ the potential behaves as $\mathcal{G} = L (\log L - 1) + \dots$

Quite generally, Kähler metrics on toric manifolds were constructed by [Guillemin \('94\)](#). They are built using Kähler quotients, and the corresponding symplectic potential exhibits the singularities just described.

Metrics on the cone and toric geometry

In our problem we have more symmetry: $U(1) \times U(2)$ instead of $U(1)^3$.
The Kähler potential is

$$\mathcal{K} = \mathcal{K} \left(\underbrace{|w|^2}_{=e^t}, \underbrace{|z_1|^2 + |z_2|^2}_{=e^s} \right),$$

which means that the metric is of cohomogeneity-2. For \mathcal{G} this implies the following form:

$$\mathcal{G} = \left(\frac{\mu}{2} + \tau \right) \log \left(\frac{\mu}{2} + \tau \right) + \left(\frac{\mu}{2} - \tau \right) \log \left(\frac{\mu}{2} - \tau \right) - \mu \log \mu + G(\mu, \nu)$$

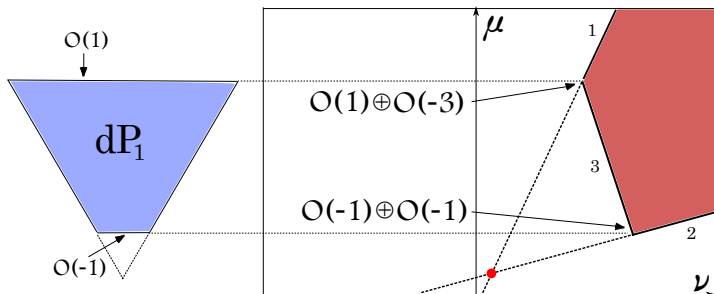
$$\mu = \mu_1 + \mu_2, \quad \tau = \frac{\mu_1 - \mu_2}{2}, \quad \nu = \mu_3.$$

Metrics on the cone and toric geometry

The Ricci-flatness equation is then a Monge-Ampère equation in two variables:

$$e^{G_\mu + G_\nu} \left(G_{\mu\mu} G_{\nu\nu} - G_{\mu\nu}^2 \right) = \mu$$

The domain of definition is the moment polytope of the cone \mathcal{M} :



The asymptotic behavior of G

One can construct an **exact** solution of the above equation taking the conical ansatz for the metric $ds^2 = dr^2 + r^2 \widetilde{ds}^2$. We make a change of variables $(\mu, \nu) \rightarrow (\nu, \xi = \frac{\mu}{\nu})$ and look for G in the form ($\nu \propto r^2$)

$$G = 3\nu (\log \nu - 1) + \nu P(\xi)$$

One obtains an ODE for $P(\xi)$ that can be solved exactly. As a result,

$$G = \sum_{i=0}^2 \frac{\mu - \xi_i \nu}{1 - \xi_i} (\log(\mu - \xi_i \nu) - 1),$$

where ξ_i are the roots of $Q(\xi) = \xi^3 - \frac{3}{2}\xi^2 + d$. Varying d , one arrives at the Sasakian manifolds called $Y^{p,q}$ discovered in [Gauntlett, Martelli, Sparks, Waldram \('2004\)](#). The topology of the underlying del Pezzo surface forces us to pick $Y^{2,1}$.

The conical metric constructed above is singular at $r = 0$. Constructing a smooth – resolved – metric is rather difficult. For the moment let us assume that, for a fixed moment polytope, we constructed one such metric with potential G_0 . To check uniqueness, one can expand $G = G_0 + H$ to first order in H :

$$\Delta_{G_0} H = 0 \quad \Rightarrow \quad 0 = \int d\mu d\nu H \Delta_{G_0} H \stackrel{?}{=} - \int d\mu d\nu (\nabla H)^2$$

Whether we may integrate by parts depends on the behavior at infinity, where we have asymptotically

$$\Delta_{G_0} H = 0 \quad \rightarrow \quad -\frac{\partial}{\partial \xi} \left(Q(\xi) \frac{\partial H}{\partial \xi} \right) + \frac{\xi}{\nu} \frac{\partial}{\partial \nu} \left(\nu^3 \frac{\partial H}{\partial \nu} \right) = 0$$

Uniqueness

Substituting $H = \nu^m h(\xi)$, we get a Heun equation

$$-\frac{d}{d\xi} \left(Q(\xi) \frac{dh}{d\xi} \right) + m(m+2) \xi h(\xi) = 0$$

Therefore one needs to estimate the spectrum of the Laplacian on $Y^{2,1}$.

We have the following result:

Proposition. [DB, 2017]

For the smallest non-zero eigenvalue λ of the Laplacian $\Delta_\xi = -\frac{d}{d\xi} \left(Q(\xi) \frac{dh}{d\xi} \right)$, entering the equation $\Delta_\xi f + \lambda \xi f = 0$, one has the lower bound $\lambda \geq 3$.

As a result, we obtain uniqueness of the metric for a given moment polytope. Therefore all potential moduli of the metric have to be related to the moduli of the polytope, which in turn are the Kähler moduli.

Part III. Killing-Yano forms.

Killing-Yano forms.

One approach to the explicit construction of a metric is to require that it admit a conformal Killing-Yano form (CKYF).

$$\nabla_i \xi_j = 0 \quad \Rightarrow \quad \text{Reduced holonomy}$$

$$\nabla_i \xi_j - \nabla_j \xi_i = 0 \quad \Rightarrow \quad \xi = d\chi$$

$$\nabla_i \xi_j + \nabla_j \xi_i = 0 \quad \Rightarrow \quad \text{Killing vector}$$

The Killing-Yano form $\omega_{ij} dx^i \wedge dx^j$:

$$\nabla_i \omega_{jk} + \nabla_j \omega_{ik} = 0$$

Conformal Killing-Yano form:

$$\nabla_i \omega_{jk} + \nabla_j \omega_{ik} - \text{trace parts} = 0$$

Killing-Yano forms.

On a Kähler manifold we may expand $\omega = \omega^{(2,0)} \oplus \omega^{(1,1)} \oplus \omega^{(0,2)}$. Especially simple is the situation when ω is Hermitian, i.e. $\omega^{(2,0)} = 0$. Introducing the 'shifted' form $\Omega_{a\bar{b}} = \omega_{a\bar{b}} - h g_{a\bar{b}}$ ($h = g^{a\bar{b}}\omega_{a\bar{b}}$), one gets the equation [Apostolov, Calderbank, Gauduchon \('2002\)](#)

$$\nabla_a \Omega_{b\bar{c}} = -2g_{a\bar{c}} \partial_b h$$

The tensor Ω has various names, such as Hamiltonian two-form, twistor form, etc. One can show that its eigenvalue functions x_i have orthogonal gradients. They can be related to 'moment map' variables μ_i corresponding to holomorphic isometries via the interesting formula:

$$\prod_{k=1}^n (\vartheta - x_k) = \sum_{k=0}^n \vartheta^k \mu_{k+1}.$$

The orthotoric metric.

At the end of the day the metric admitting a tensor Ω has the form (we set $x_1 = x, x_2 = y$, then $\mu = xy, \nu = x + y$)

$$ds^2 = xy g_{\mathbb{CP}^1} + (x - y) \left(\frac{dx^2}{P_1(x)} + \frac{dy^2}{P_2(y)} \right) + \text{angular part}$$

We call this metric the 'orthotoric metric'. We see that the variables separated. The requirement of Ricci-flatness fixes the functions P_1, P_2 to be cubic polynomials (one of which we encountered before):

$$P_1(x) = x^3 - \frac{3}{2}x^2 + c \quad P_2(y) = y^3 - \frac{3}{2}y^2 + d.$$

The domain is $x \leq x_{min}, y \in [y_1, y_2]$.

The orthotoric metric.

If we further require that the topology is that of the cone over \mathbf{dP}_1 , the constants c and d are uniquely fixed. This metric was also obtained by [Chen, Lü, Pope \('2006\)](#), [Oota, Yasui \('2006\)](#) and was extensively studied by [Martelli, Sparks \('2007\)](#).

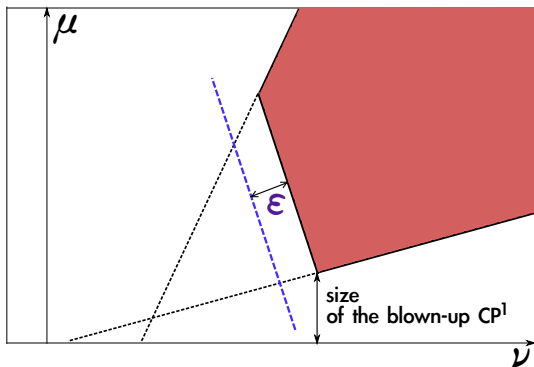
The point is that the requirements of

- (a) Ricci-flatness
 - (b) Cone over \mathbf{dP}_1 topology
 - (c) CKYF of type $(1, 1)$
- completely fix the metric.

According to the CY theorem, however, the metric should contain additional parameters, corresponding to the deformation of the moment polytope. Altogether there are 2 parameters, since $H^2(\mathcal{M}, \mathbb{R}) = \mathbb{R}^2$.

Deformation of the metric and the CKYF.

One parameter is somewhat 'trivial', as it corresponds to a rescaling of the metric. We can still look for the other non-trivial parameter, which corresponds to the following deformation:



Deformation of the metric and the CKYF.

In the equation $\Delta_{G_0} H = 0$, if we substitute the orthotoric potential G_0 , variables separate:

$$\frac{1}{x} \frac{\partial}{\partial x} \left(P_1(x) \frac{\partial H}{\partial x} \right) - \frac{1}{y} \frac{\partial}{\partial y} \left(P_2(y) \frac{\partial H}{\partial y} \right) = 0$$

The unique solution compatible with the deformation of the moment polytope is

$$H(x, y) = \epsilon \int_x^\infty \frac{d\hat{x}}{P_1(\hat{x})}.$$

For large x one has $H(x, y) = \frac{\epsilon}{2x^2} + \dots$, and for the metric this implies $|g - g_0|_{g_0} = O\left(\frac{1}{r^6}\right)$. This implies that the variation of the Kähler form has the property $[\delta\omega] \in H_c^2(\mathcal{M}, \mathbb{R})$.

Deformation of the metric and the CKYF.

The next question is: what happens to the Killing-Yano form?

If it is deformed, it must acquire a non-zero $(2,0)$ part, i.e. $\omega^{2,0} = \omega_{mn} dz^m \wedge dz^n \neq 0$. On a Calabi-Yau manifold, one has a nowhere vanishing three-form $\Omega_{mnp} dz^m \wedge dz^n \wedge dz^p$, and one can use it to dualize $\omega^{2,0}$ and obtain a vector field $\omega^p := \tilde{\Omega}^{mnp} \omega_{mn}$.

Using that \mathcal{M} is Ricci-flat and assuming that all Killing vector fields on \mathcal{M} are holomorphic, we can show that ω^p has to satisfy a rather stringent requirement

$$R^n_{mp\bar{k}} \omega^p = 0. \quad (1)$$

Deformation of the metric and the CKYF.

As we mentioned earlier, on a toric manifold the curvature tensor is $R_{\bar{m}jk\bar{n}} = -\sum_{s,t} \mathcal{G}_{ns}^{-1} \frac{\partial^2 \mathcal{G}_{jk}^{-1}}{\partial \mu_s \partial \mu_t} \mathcal{G}_{tm}^{-1}$. Using the explicit expression for the orthotoric potential \mathcal{G} , we can show that the only solution is $\omega^p = 0$.

Proposition. [DB, 2017]

There exists a first-order deformation of the orthotoric metric that preserves Ricci-flatness and corresponds to a deformation of the moment polytope. Moreover, the deformation of the Kähler form has the property $[\delta\omega] \in H_c^2(\mathcal{M}, \mathbb{R})$. The deformed metric does not possess a conformal Killing-Yano tensor.

Summary.

- Metrics on non-compact Calabi-Yau manifold can be sometimes constructed explicitly
- Examples in $\dim_{\mathbb{C}} \mathcal{M} = 3$: Cones over $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$
- More complicated cases with conformal Killing-Yano tensors
- In the case of the cone over dP_1 the corresponding metric is not the most general one, predicted by the CY theorem
- One can explicitly construct a first-order deformation
- What is the significance of the explicitly known (orthotoric) metric? Can one obtain a closed expression for the metric in the general case, or in other special cases?