Mirror and spectral dualities in $3d$ $T[SU(N)]$ theories

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based on arXiv:1712.08140 with Nedelin and Pasquetti
and work in progress with Aprile, Nedelin, Nieri, Pasquetti, Sacchi

Introduction and plan

Why are we interested in the zoo of obscure $3d$ SUSY theories? Because they are interconnected by interesting dualities. According to the relative paradigm:

Objects are secondary, while the relations between them are fundamental.

The theories we are going to consider are also related to topological strings and $5d$ gauge theories.

Plan

- $3d$ $\mathcal{N} = 2$ quiver theories, mirror and spectral dualities and what is the difference between the two
- $3d$ spectral dualities from $5d$ and topological strings via Higgsing/geometric transition

3d $\mathcal{N} = 2$ basics

Field content:

- Vector multiplets: $V = (A_\mu, \lambda, \sigma \in \mathbb{R}, aux)$
- Adjoint chiral multiplets: $\Phi = (\phi \in \mathbb{C}, fermions, aux)$
- Matter chiral multiplets (usually in fundamental rep of the gauge group): $Q_i = (Q_i \in \mathbb{C}, fermions, aux)$
- One can also introduce the linear multiplets: $\Sigma = (\sigma, .., F_{\mu\nu})$.

The moduli space of vacua:

Pure Higgs branch where $\langle Q_i \rangle \neq 0$ and $\langle \sigma \rangle = 0$ and G is partly broken, e.g. $SU(N) \to SU(N-1)$

Pure Coulomb branch where $\langle Q_i \rangle = 0$ and $\langle \sigma \rangle \neq 0$, which breaks the gauge group to its Cartan: $G \to U(1)^r$

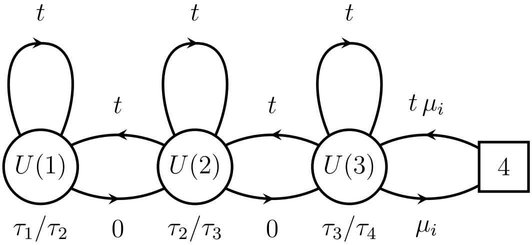
Mixed branches

In the bulk of the (abelianised) Coulomb branch one can dualise the gauge fields to scalars: $F^j_{\mu\nu} = \epsilon_{\mu\nu\rho}\partial^\rho\gamma_j\,,\ j = 1, \cdots r$.

The currents $J^j_\mu = \epsilon_{\mu\nu\rho}(F^{\nu\rho})^j$ generate the topological symmetry $(U(1)_J)^r$ which shifts the dual photons $\gamma_j \to \gamma_j + \alpha_j$.

There is also flavor symmetry $SU(N_f)$ acting on the matter multiplets: $Q_i \to \Omega^j_i Q_j$

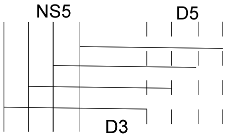
Example:

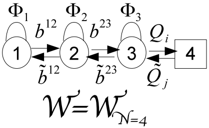


where and $\mu_p = e^{RM_p}$, $\tau_p = e^{RT_p}$, $t = q^{\frac{1}{2}}e^{Rm_A}$.

$T[SU(N)]$

The $\mathcal{N} = 4$ $T[SU(N)]$ is a quiver theory introduced as boundary condition for 4d SYM [Gaiotto-Witten]. Low energy theory on D3 branes suspended between NS5 and D5:





	0	1	2	3	4	5	6	7	8	9
NS5	—	—	—					—	—	—
D5	—	—	—		—	—	—			
D3	—	—	—	—						
D5′	—	—	—				—		—	—

- Global symmetry: $SU(N)_F \times SU(N)_{top}$
- Self-dual under mirror symmetry: Coulomb $\leftrightarrow$ Higgs branch
- Masses M_p, T_p in $SU(N)_F \times SU(N)_{top}$
- Axial mass $m_A \in SU(2)_C \times SU(2)_H$ breaking to $\mathcal{N} = 2^*$
- The mass deformed theory has $N!$ isolated vacua

$T[SU(N)]$ and its mirror dual

The gauge invariant operators include the mesons:

$$\mathbb{Q}_{ij} \equiv Q_i \tilde{Q}_j \,,$$

and the monopole operators:

$$\mathcal{M}_{ij} \equiv \begin{pmatrix} \mathrm{Tr}\Phi^{(1)} & \mathcal{M}^{100} & \mathcal{M}^{110} & \mathcal{M}^{111} \\ \mathcal{M}^{-100} & \mathrm{Tr}\Phi^{(2)} & \mathcal{M}^{010} & \mathcal{M}^{011} \\ \dots & \dots & \dots & \dots \end{pmatrix} .$$

The mirror dual theory $\check{T}[SU(N)]$ has gauge invariant operators $\check{\mathcal{M}}_{ij}$ and $\check{\mathbb{Q}}_{ij}$.

Operator map:

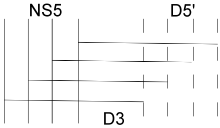
$$\mathbb{Q}_{ij} \leftrightarrow \check{\mathcal{M}}_{ij} \,, \qquad \mathcal{M}_{ij} \leftrightarrow \check{\mathbb{Q}}_{ij}$$

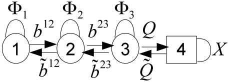
The partition functions satisfy:

$$Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A) = Z_{T[SU(N)]}(\vec{T}, \vec{M}, -m_A) \,.$$

$FT[SU(N)]$ and its dual

We take *Fourier* transform in the mesons using N^2 gauge singlets X_{ij} and obtain $FT[SU(N)]$ describing D3 branes between NS5 and D5':





$$\mathcal{W} = \mathcal{W}_{\mathcal{N}=4} + X_{ij} Q_i \tilde{Q}_j$$

	0	1	2	3	4	5	6	7	8	9
NS5	–	–	–					–	–	–
D5	–	–	–		–	–	–			
D3	–	–	–	–						
D5′	–	–	–				–		–	–

Now the gauge invariant operators are $\mathcal{M}_{ij}$, X_{ij}. The dual theory is $\check{FT}[SU(N)]$ with the operator map

$$X_{ij} \leftrightarrow \check{\mathcal{M}}_{ij}\,, \qquad \mathcal{M}_{ij} \leftrightarrow \check{X}_{ij}$$

The partition functions satisfy:

$$Z_{FT[SU(N)]}(\vec{M}, \vec{T}, m_A) = Z_{FT[SU(N)]}(\vec{T}, \vec{M}, m_A)$$

Notice that $Z_{FT[SU(N)]}(\vec{M}, \vec{T}, m_A) = K[\vec{M}, m_A] Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A)$ with $K[\vec{M}, m_A]$ the contribution of the singlets X_{ij} in the $SU(N)$ adjoint.

Difference operators

It has been shown that the $T[SU(N)]$ partition function is an eigenfunction of the N-body trigonometric Ruijsenaars-Schneider (RS) Hamiltonians [Gaiotto-Koroteev],[Bullimore-Kim-Koroteev]:

$$T_r(\vec{M}, m_a) Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A) = \chi_r(\vec{T}) Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A)$$

$$T_r(\vec{T}, -m_a) Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A) = \chi_r(\vec{M}) Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A)\,,$$

with $r = 1, \cdots, N$. This has been used to derive the identity for mirror self-duality:

$$Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A) = Z_{T[SU(N)]}(\vec{T}, \vec{M}, -m_A)\,.$$

Moreover since $T_r(\vec{M}, -m_a) = K[\vec{M}, m_A]^{-1} T_r(\vec{M}, m_a) K[\vec{M}, m_A]$ we can also prove the identity for spectral self-duality

$$Z_{FT[SU(N)]}(\vec{M}, \vec{T}, m_A) = Z_{FT[SU(N)]}(\vec{T}, \vec{M}, m_A)\,.$$

Bonus duality

Using the difference operators we get another identity:

$$Z_{T[SU(N)]}(\vec{M}, \vec{T}, m_A) = K[\vec{M}, m_A]^{-1} K[\vec{T}, m_A] Z_{T[SU(N)]}(\vec{M}, \vec{T}, -m_A).$$

This is actually a new duality (it reduces to the known Aharony duality in the $N = 2$ case).

The l.h.s. theory is $T[SU(N)]$ with $\mathcal{W} = \mathcal{W}_{\mathcal{N}=4}$ with operators including monopoles $\mathcal{M}_{ij}$ and mesons $\mathbb{Q}_{ij}$.

The r.h.s. theory is $FFT[SU(N)]$ with $\mathcal{W} = \mathcal{W}_{\mathcal{N}=4} + S_{ij}\mathcal{M}_{ij} + X_{ij} Q_i Q_j$. The gauge invariant operators in $FFT[SU(N)]$ include gauge singlets:

$$X_{ij} , \qquad S_{ij}.$$

Operator map:

$$\mathbb{Q}_{ij} \to X_{ij} , \qquad \mathcal{M}_{ij} \to S_{ij} .$$

$T[SU(N)]$ holomorphic block integral

We will work with $D_2 \times S^1$ partition functions, the holomorphic blocks.

$$\mathcal{B}^{D_2\times S^1,(\alpha_0)}_{T[SU(N)]} = Z^{3d,(\alpha_0)}_{\mathrm{cl}} Z^{3d,(\alpha_0)}_{1-\mathrm{loop}} Z^{3d,(\alpha_0)}_{\mathrm{vort}} ,$$

$$Z^{3d,(\alpha_0)}_{\mathrm{vort}}(\vec{\mu},\vec{\tau},q,t) =$$

$$= \sum_{\{k_i^{(a)}\}} \prod_{a=1}^{N-1} \left[\left(t\frac{\tau_a}{\tau_{a+1}} \right)^{\sum_{i=1}^{a} k_i^{(a)}} \prod_{i\neq j}^{a} \frac{\left(t\frac{\mu_i}{\mu_j};q\right)_{k_i^{(a)}-k_j^{(a)}}}{\left(\frac{\mu_i}{\mu_j};q\right)_{k_i^{(a)}-k_j^{(a)}}} \prod_{i=1}^{a}\prod_{j=1}^{a+1} \frac{\left(\frac{q}{t}\frac{\mu_i}{\mu_j};q\right)_{k_i^{(a)}-k_j^{(a+1)}}}{\left(q\frac{\mu_i}{\mu_j};q\right)_{k_i^{(a)}-k_j^{(a+1)}}} \right]$$

the sum is over sets of integers $k_i^{(a)}$ satisfying the inequalities

$$\begin{array}{ccccc} k_1^{(1)} \geq & k_1^{(2)} \geq & k_1^{(3)} \geq & \cdots \geq & k_1^{(N-1)} \geq 0 \\ & k_2^{(2)} \geq & k_2^{(3)} \geq & \cdots \geq & k_2^{(N-1)} \geq 0 \\ & & & \ddots & \vdots \\ & & & & k_{N-1}^{(N-1)} \geq 0 \end{array}$$

Duality for the blocks:

$$\mathcal{B}^{D_2\times S^1}_{FT[SU(N)]}(\vec{\mu},\vec{\tau},t) = \mathcal{B}^{D_2\times S^1}_{FT[SU(N)]}(\vec{\tau},\vec{\mu},t),$$
$$\mathcal{B}^{D_2\times S^1}_{T[SU(N)]}(\vec{\mu},\vec{\tau},t) = \mathcal{B}^{D_2\times S^1}_{T[SU(N)]}(\vec{\tau},\vec{\mu},\tfrac{q}{t}) .$$

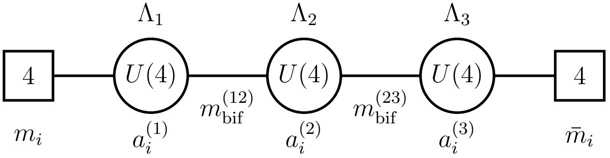
$3d$ $FT[SU(N)]$ and its dual from $5d$

$FT[SU(N)]$ lives on D3 branes suspended between N NS5s and N D5's. These branes form the (p,q)-web engineering the $5d$ $\mathcal{N}=1$ quiver theory $N+SU(N)^{N-1}+N$.

We want to view $FT[SU(N)]$ as a codimension-two defect in this theory:

- Higgsing: the $FT[U(N)]$ partition function is obtained by tuning the parameters of the $5d$ square quiver partition function.
- Brane realisation: the codimension-two defect theory is the vortex string theory on the Higgs branch of the $5d$ theory.
- Geometric engineering: Higgsing corresponds to geometric transition happening at quantised values of the Kähler parameters.
- 3d spectral duality descends from fiber-base or IIB S-duality

Higgsing the $5d$ square quiver



The instanton partition function $Z^{5d}_{inst}[U(N)^{N-1}]$ is a sum over N-tuples of Young diagrams, $\vec{Y}^{(a)} = \{Y_1^{(a)}, \ldots, Y_N^{(a)}\}$, $a = 1, \ldots, (N-1)$.

When the Coulomb branch parameters are tuned to special values, the Young diagrams for some nodes truncate to diagrams with finitely many columns yielding the partition function of a coupled system:

$$Z^{5d}[U(N)^{N-1}] \xrightarrow[\text{Higgsing}]{} Z^{5d-3d} \, .$$

For *maximal* Higgsing the $5d$ bulk theory is trivial and we just get the vortex partition function of the $3d$ theory.

$FT[SU(N)]$ via Higgsing

By *maximally* Higgsing the $5d$ square quiver by tuning masses and Coulomb parameters as:

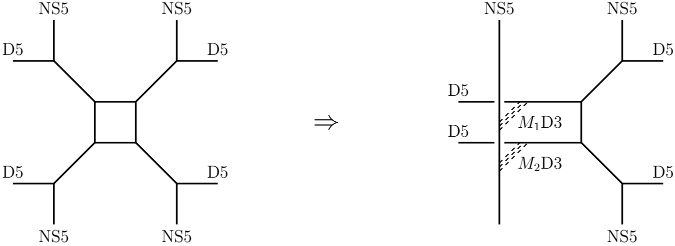
$$\begin{aligned} a_1^{(1)} &= m_1 t, \\ a_2^{(1)} &= m_2, \\ &\vdots \\ a_{N-1}^{(1)} &= m_{N-1}, \\ a_N^{(1)} &= m_N, \end{aligned} \qquad \cdots \qquad \begin{aligned} a_1^{(N-1)} &= m_1 t, \\ a_2^{(N-1)} &= m_2 t, \\ &\vdots \\ a_{N-1}^{(N-1)} &= m_{N-1} t, \\ a_N^{(N-1)} &= m_N, \end{aligned} \qquad \begin{aligned} \bar{m}_1 &= m_1 \tfrac{t^2}{q}, \\ \bar{m}_2 &= m_2 \tfrac{t^2}{q}, \\ &\vdots \\ \bar{m}_{N-1} &= m_{N-1} \tfrac{t^2}{q}, \\ \bar{m}_N &= m_N \tfrac{t^2}{q} \end{aligned}$$

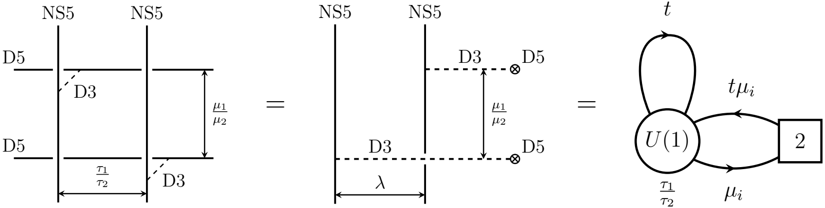
we obtain $FT[SU(N)]$:

$$Z^{5d}[U(N)^{N-1}] \to \mathcal{B}_{FT[SU(N)]}^{D_2 \times S^1, (\alpha_0)} .$$

Higgsing and branes

The 5d square quiver can be realised as the low energy description of a web of N NS5 and N D5' branes.

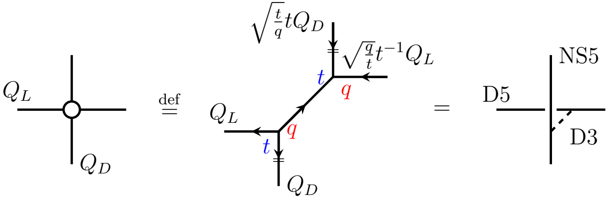




On the Higgs branch the NS5 branes can be removed from the web and D3 stretched in between. The $3d$ low energy theory on the D3s is our vortex theory.

Higgsing and geometric transition

Equivalently we can engineer the 5d quiver theory from M-theory on $X \times \mathbb{R}^4_{q,t} \times S^1$ with X toric CY 3-fold, which can be drawn as a toric diagram. The Higgsing conditions translate into quantisation condition for Kähler parameters $Q = \sqrt{\frac{q}{t}} t^N$ for which there is geometric transition.

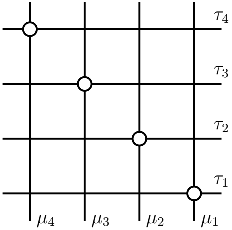


Using the refined topological vertex technique one can check that:

$$Z^X_{\text{top}}(\vec{\mu}, \vec{\tau}, q, t) = \mathcal{B}^{D_2 \times S^1, (\alpha_0)}_{FT[SU(N)]}(\vec{\mu}, \vec{\tau}, q, t)$$

$\vec{\mu}, \vec{\tau}$ are identified with fiber and base Kähler paramters.

$3d$ duality from fiber-base duality



The CY X is invariant under the action fiber-base duality which swaps μ_i with τ_i and so

$$Z_{\rm top}^{X}(\vec{\mu},\vec{\tau},q,t) = Z_{\rm top}^{X}(\vec{\tau},\vec{\mu},q,t)\,.$$

Notice that t is an Ω-background parameter not affected by the map, as in

$$\mathcal{B}^{D_2\times S^1,\,(\alpha)}_{FT[SU(N)]}(\vec{\mu},\vec{\tau},t) = \mathcal{B}^{D_2\times S^1,\,(\alpha)}_{FT[SU(N)]}(\vec{\tau},\vec{\mu},t)\,.$$

$\rightarrow$ 3d self-duality for $FT[SU(N)]$ descends from fiber-base duality!

We can generate large families of new 3d dualities from fiber-base via Higgsing. New quantitative approach to study 3d dualities!

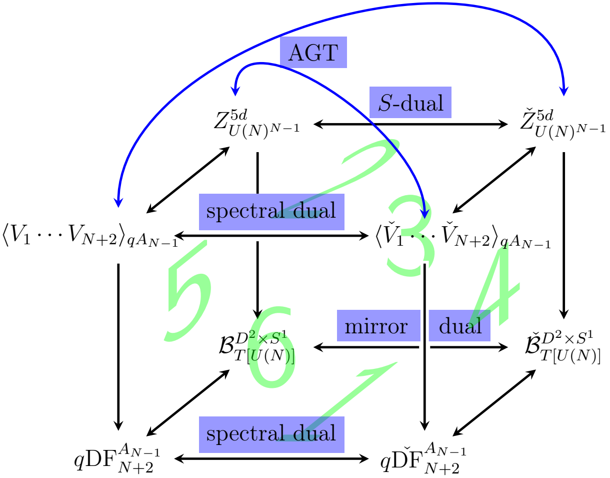
Conclusions

- There are two distinct mirror-like dualities in $3d$ $\mathcal{N} = 2$ gauge theories: one inverting the $\mathcal{N} = 2$ deformation parameter, the other keeping it fixed.
- These theories can be understood from the $5d$ point of view using the Higgsing procedure.
- One of the mirror-like dualities is explained by the spectral duality of refined topological string.
- The other duality gives a new relation for the amplitudes of the refined topological string, geometrically engineering the $5d$ gauge theory.
- There are many more relations — e.g. with conformal blocks of q-Virasoro, ordinary Virasoro and exotic d-Virasoro algebras.

Thank you for your attention!

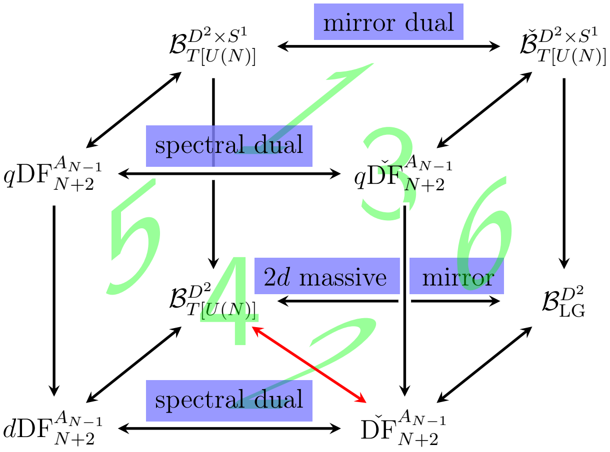
What was left aside

3d gauge theory partition functions as q-Toda blocks



What was left aside

$3d \to 2d$ compactifications and $q \to 1$ limits: several of them



What was left aside

d-Virasoro algebra

Starting from the q-Virasoro relation:

$$f\left(\frac{w}{z}\right)T(z)T(w) - f\left(\frac{z}{w}\right)T(w)T(z) = -\frac{(1-q)(1-t^{-1})}{1-\frac{q}{t}}\left(\delta\left(\frac{q}{t}\frac{w}{z}\right) - \delta\left(\frac{t}{q}\frac{w}{z}\right)\right)$$

We can take an unconventional limit where $z = q^u, w = q^v$ and finite $t(u) = \lim_{q\to 1} T(q^u)$ to obtain the d-Virasoro algebra:

$$g(v-u)t(u)t(v) - g(u-v)t(v)t(u) = \frac{\beta}{\beta-1}\left(\delta(v-u+1-\beta) - \delta(v-u-1+\beta)\right)$$

with

$$g(u) = \frac{2(1-\beta)}{u}\frac{\Gamma\left(\frac{u+2-\beta}{2(1-\beta)}\right)\Gamma\left(\frac{u+1-2\beta}{2(1-\beta)}\right)}{\Gamma\left(\frac{u+1}{2(1-\beta)}\right)\Gamma\left(\frac{u-\beta}{2(1-\beta)}\right)}.$$

We found a bosonization of this algebra: the screening current and vertex operators and calculated their normal ordered correlators $dDF_{N+2}^{A_{N-1}}$. They are equal to ordinary CFT correlators!