A non-torus link from topological vertex

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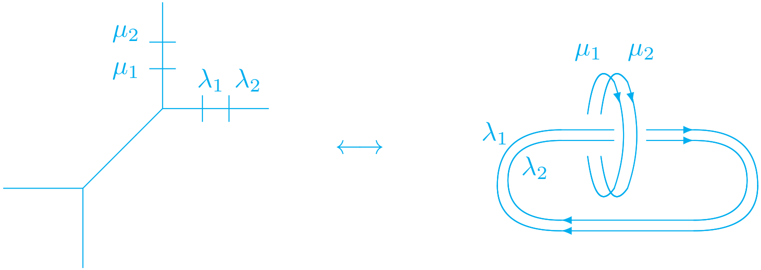
Basic statement



$$\mathcal{G}^{L_{8n8}}_{\mu_1\times\lambda_1\times\mu_2\times\lambda_2} := \mathbf{Pr}\left[\mathcal{H}^{L_{8n8}}_{\mu_1\times\lambda_1\times\mu_2\times\lambda_2}\right]_{\max} = \frac{Z_{\mu_1,\mu_2;\lambda_1,\lambda_2}}{Z_{\varnothing,\varnothing:\varnothing,\varnothing}} = \mathcal{H}^{\mathrm{Hopf}}_{(\lambda_1,\lambda_2)\times(\mu_1,\mu_2)}$$

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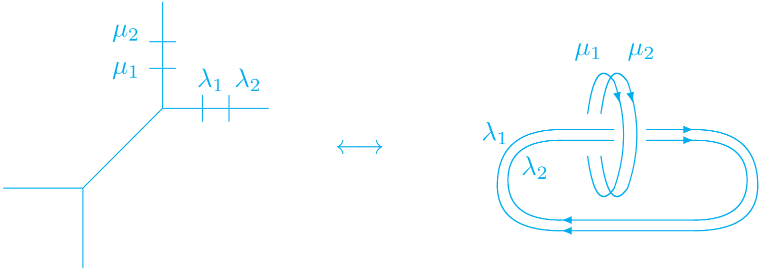

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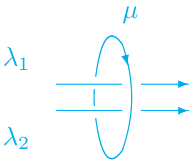
Hopf polynomial as a character



This tangle can be treated in two different ways:

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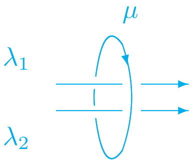

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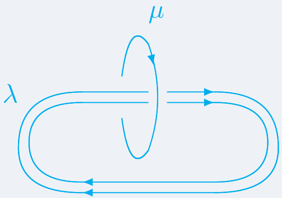

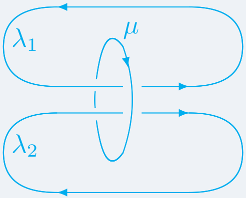

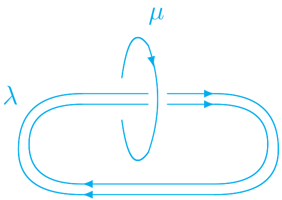

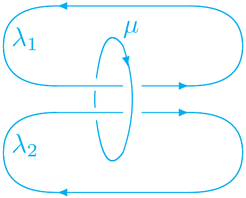

$$D_\mu \cdot \sum_{\lambda \in \lambda_1 \otimes \lambda_2} N^{\lambda}_{\lambda_1 \lambda_2} \cdot \mathcal{H}^{\rm Hopf}_{\lambda,\mu} \;=\; \mathcal{H}^{\rm Hopf}_{\lambda_1,\mu} \cdot \mathcal{H}^{\rm Hopf}_{\lambda_2,\mu}$$

$$\text{Schur}_{\lambda_1} \cdot \text{Schur}_{\lambda_2} = \sum_{\mu} N^{\mu}_{\lambda_1 \lambda_2} \ \text{Schur}_{\mu}$$

$$\mathcal{H}^{\rm Hopf}_{\lambda,\mu} = \underbrace{{\rm Schur}_\lambda(q^{-\rho})}_{D_\lambda} \cdot {\rm Schur}_\mu(q^{-\lambda-\rho})$$

Time variables $p_k := \sum_i x_i^{2k}$, $A = q^N$:

$$D_\mu = {\rm Schur}_\mu\{p^*\}, \qquad p_k^* := \frac{A^k - A^{-k}}{q^k - q^{-k}}$$

$$p_k^{*\lambda} = p_k^* - A^{-k}(q^k - q^{-k}) \sum_{i,j\in\lambda} q^{2k(i-j)} = p_k^* + A^{-k} \sum_i q^{(2i-1)k}(q^{-2k\lambda_i} - 1)$$

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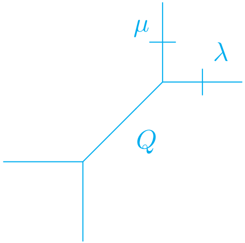
Conifold description



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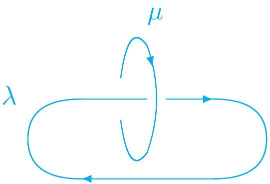

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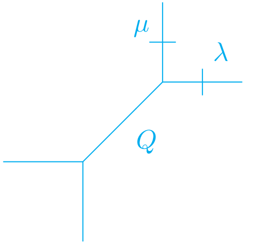

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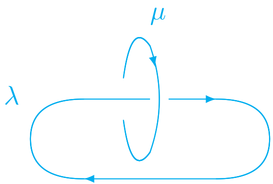

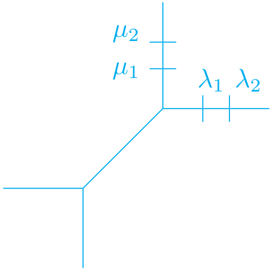

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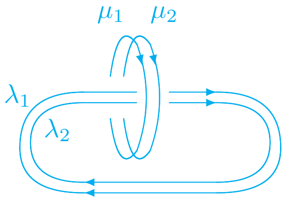
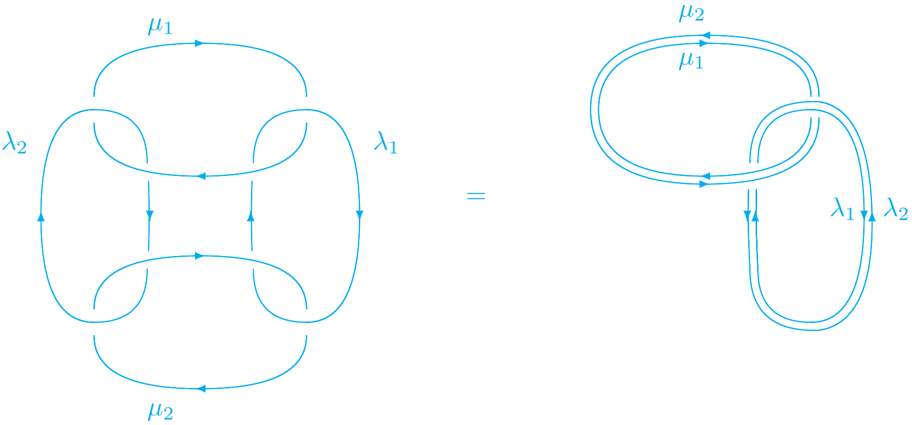
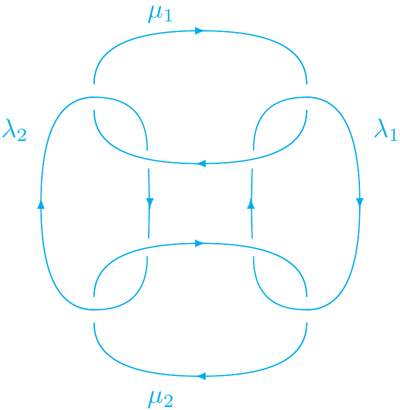




$$\mathcal{H}^{L_{8n8}}_{\mu_1,\lambda_1,\mu_2,\lambda_2} = \sum_{\substack{\lambda\in\lambda_1\otimes\bar\lambda_2 \\ \mu\in\mu_1\otimes\bar\mu_2}} N^{\lambda}_{\lambda_1\lambda_2}\cdot N^{\mu}_{\mu_1\mu_2}\cdot \mathcal{H}^{\text{Hopf}}_{\lambda,\mu}$$



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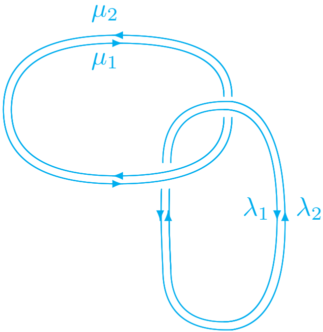

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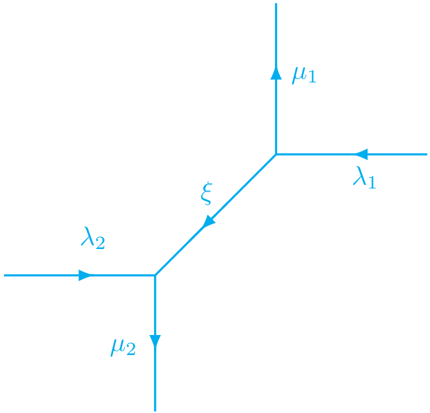
Hopf link versus L8n8



Topological vertex:

$$C_{\xi\mu\lambda}(q) = q^{\varkappa(\lambda)} \cdot \mathrm{Schur}_\mu(q^{\rho_0}) \sum_\eta \mathrm{Schur}_{\xi/\eta}(q^{\mu+\rho_0}) \cdot \mathrm{Schur}_{\lambda^\vee/\eta}(q^{\mu^\vee+\rho_0}),$$

$$p_k^{(\mu)} = p_k(q^{\mu+\rho_0}) = \frac{1}{q^k - q^{-k}} + \sum_{j=1}^{\infty} q^{(1-2j)k}(q^{2\mu_j k} - 1)$$

$$\varkappa(\lambda) = 2\sum_{i,j\in\lambda}(j-i), \qquad 2\rho = (N-1, N-3, \ldots, -N+3, -N+1) \qquad 2\rho_0 = (-1, -3, \ldots)$$

Four point function" on the resolved conifold geometry:

$$Z_{\mu_1,\mu_2;\lambda_1,\lambda_2} = \sum_\xi (-Q)^{|\xi|} C_{\xi\mu_1^\vee\lambda_1}(q) C_{\xi^\vee\mu_2\vee\lambda_2}(q),$$

$Q = e^t$, t is the Kähler parameter of the rational curve $\mathbf{P}^1$ represented by the internal edge. Large N duality: the 't Hooft coupling is $t = Ng_s = \frac{2\pi i N}{N+k}$ hence, $Q = q^{2N} = A^2$.

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Summation using the Cauchy formula for the skew Schur functions:

$$\sum_{\xi}(-Q)^{|\xi|}\cdot \mathrm{Schur}_{\xi/\eta_1}\{p\}\cdot \mathrm{Schur}_{\xi^\vee/\eta_2}\{p'\} =$$

$$= \quad \exp\left(-\sum_k \frac{Q^k p_k p'_k}{k}\right)\cdot \sum_{\sigma}(-Q)^{|\eta_1|+|\eta_2|-|\sigma|}\cdot \mathrm{Schur}_{\eta_1^\vee/\sigma}\{p'\}\cdot \mathrm{Schur}_{\eta_2^\vee/\sigma^\vee}\{p\}$$

where

$$\exp\left(-\sum_k \frac{Q^k\, p_k^{(\mu_1)} p_k^{(\mu_2)}}{k}\right) = (-A)^{|\mu_1|+|\mu_2|} q^{\frac{\varkappa(\mu_1)+\varkappa(\mu_2)}{2}} h_{\mu_1} h_{\mu_2} D_{(\mu_1,\mu_2)} \prod_{i=1}^{\infty}(1-Qq^{-2i})^i$$

The main claims:

$$\frac{Z_{\mu_1,\mu_2;\lambda_1,\lambda_2}}{Z_{\varnothing,\varnothing:\varnothing,\varnothing}} = (-A)^{|\mu_1|+|\mu_2|} q^{\varkappa(\lambda_1)+\varkappa(\lambda_2)+\varkappa(\mu_1)+\varkappa(\mu_2)} \cdot D_{(\mu_1,\mu_2)} \times$$

$$\sum_{\sigma,\eta_1,\eta_2} (-A^2)^{|\eta_1|+|\eta_2|-|\sigma|} \cdot \mathrm{Schur}_{\lambda_1^\vee/\eta_1}\{p^{(\mu_1^\vee)}\} \cdot \mathrm{Schur}_{\lambda_2^\vee/\eta_2}\{p^{(\mu_2^\vee)}\} \cdot \mathrm{Schur}_{\eta_1^\vee/\sigma}\{p^{(\mu_2)}\} \cdot \mathrm{Schur}_{\eta_2^\vee/\sigma^\vee}\{p^{\mu_1}\}$$

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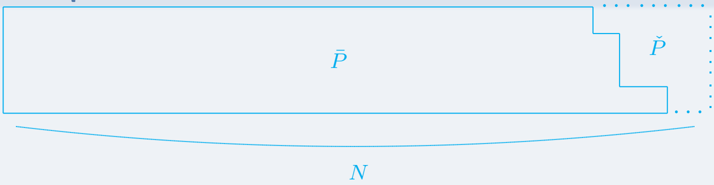
Hopf in the composite representation

Conjugate representation



$$R \otimes \bar{P} = (R, P) \oplus \ldots$$

$(R, P) \;=$

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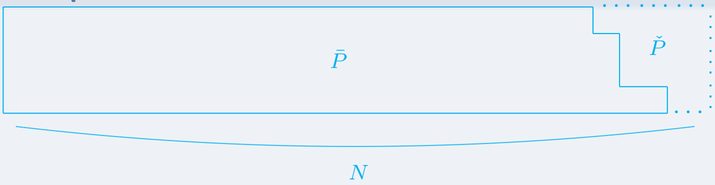

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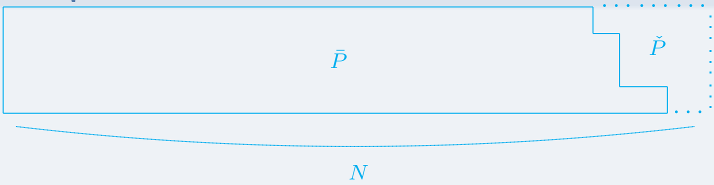

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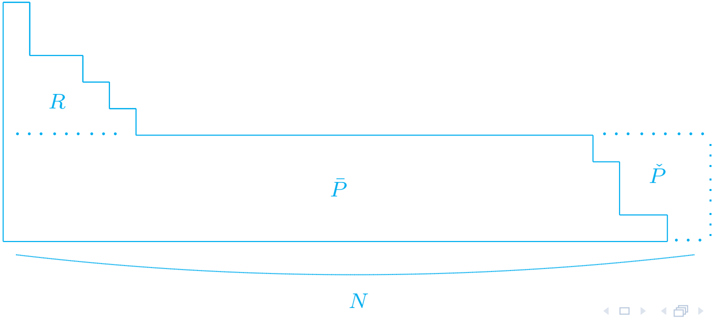

$$\mathcal{H}_{(R,P)\times(Q,S)} = D_{(R,P)} \cdot \mathrm{Schur}_{(Q,S)}\{p^{*(R,P)}\}$$

with

$$p_k^{*(R,P)} = \frac{A^k - A^{-k}}{q^k - q^{-k}} + \frac{1}{A^k} \cdot \sum_{j=1}^{l_R} q^{(2j-1)k} \cdot (q^{-2kr_j} - 1) + A^k \cdot \sum_{i=1}^{l_P} q^{(1-2i)k} \cdot (q^{2kp_i} - 1)$$

$$\mathrm{Schur}_{(R,P)}\{p^{*(Q,S)}\} = \sum_{\eta \in R \cap P^\vee} (-)^{|\eta|} \cdot \mathrm{Schur}_{R/\eta}\{p^{*(Q,S)}\} \cdot \mathrm{Schur}_{P/\eta^\vee}\{p^{*(Q,S)}(A^{-1}, q^{-1})\}$$

$\mathrm{Schur}_{R/\eta}$ are the skew Schur function. Mirror reflection $(A,q) \to (A^{-1}, q^{-1})$ could be replaced just by transposition of Young diagrams.

Positive/negative A-mode decomposition:

$$p_k^{*(R,P)} = A^k \left(\frac{1}{q^k - q^{-k}} + \sum_{i=1}^{l_P} q^{(1-2i)k} \cdot (q^{2kp_i} - 1) \right) + A^{-k} \left(-\frac{1}{q^k - q^{-k}} + \sum_{j=1}^{l_R} q^{(2j-1)k} \cdot (q^{-2kr_j} - 1) \right)$$

$$\mathrm{Schur}_{R/Q}\{p^{(1)} + p^{(2)}\} = \sum_P \mathrm{Schur}_{R/P}\{p^{(1)}\} \cdot \mathrm{Schur}_{P/Q}\{p^{(2)}\}$$

$$A^{|R|-|Q|}\mathrm{Schur}_{R^\vee/Q^\vee}\{p_k\} = \mathrm{Schur}_{R/Q}\{A^k p_k\}, \qquad p_k^{(\mu_1^\vee)} = (-1)^{k+1} p_k^{(\mu_1)}\Big|_{q\to -1/q}$$

$$\sum_{\eta_1} (-A^2)^{|\eta_1|} \cdot \mathrm{Schur}_{\lambda_1^\vee/\eta_1}\{p^{(\mu_1^\vee)}\} \cdot \mathrm{Schur}_{\eta_1^\vee/\sigma}\{p^{(\mu_2)}\} = A^{|\sigma|}(-A)^{-|\lambda_1|} \cdot \mathrm{Schur}_{\lambda_1/\sigma}\{p^{*(\mu_1,\mu_2)}\}$$

Rosso-Jones formula

$$\mathcal{H}^{\rm Hopf}_{\lambda\times\mu} = q^{\varkappa_\lambda+\varkappa_\mu}\sum_{\eta\in\lambda\otimes\mu} N^{\eta}_{\lambda\mu}\cdot q^{-\varkappa_\eta}\cdot D_\eta$$

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Schur representation $\longleftarrow$ Averaging procedure

$\nwarrow$ $\nearrow$

$\updownarrow$ **Hopf polynomial** $\downarrow$

$\swarrow$ $\searrow$

Topological vertex approach Rosso-Jones formula

Refinement

Refined Rosso-Jones formula:

$$\boxed{\mathcal{P}^{\text{Hopf}}_{\lambda,\mu} = q^{-\nu_\lambda-\nu_\mu}t^{\nu'_\lambda+\nu'_\mu}\sum_{\eta\in\lambda\otimes\mu}\mathfrak{N}^{\eta}_{\lambda\mu}\cdot q^{\nu_\eta}t^{-\nu'_\eta}\cdot\mathcal{M}_\eta}$$

where $\nu_\lambda := 2\sum_i(i-1)\lambda_i$, $\nu'_\lambda := \nu_{\lambda^\vee}$, i.e. $\varkappa_\lambda = \nu'_\lambda - \nu_\lambda$, M_η is the Macdonald dimension of η, i.e. the specialization of the Macdonald symmetric function $M_\eta\{q,t|p\}$ at the topological locus (in time variables) $p_k = p^*_k$ [?]:

$$\mathcal{M}_\eta := M_\eta\{q,t|\mathfrak{p}^*\}, \qquad \mathfrak{p}^*_k = \frac{A^k - A^{-k}}{t^k - t^{-k}}, \qquad M_\lambda\{q,t|p\}\cdot M_\mu\{q,t|p\} = \sum_{\eta\in\lambda\otimes\mu}\mathfrak{N}^{\eta}_{\lambda\mu}\cdot M_\eta\{q,t|p\}$$

$\mathfrak{N}^{\eta}_{\lambda\mu}$ are not obligatory integer.

$$\mathcal{P}^{\text{Hopf}}_{\lambda,\mu} = M_\lambda(t^{-\rho}) \cdot M_\mu(q^{-\lambda}t^{-\rho}) = \mathcal{M}_\lambda \cdot M_\mu(q^{-\lambda}t^{-\rho})$$

These symmetric functions of the components of vectors in the Cartan plane can be again rewritten in terms of the time variables

$$\mathfrak{p}^{*\lambda}_k = \mathfrak{p}^*_k - A^{-k}(q^k - q^{-k}) \sum_{i,j\in\lambda} t^{k(2i-1)} q^{k(1-2j)} = \mathfrak{p}^*_k + A^{-k} \sum_i t^{(2i-1)k}(q^{-2k\lambda_i} - 1)$$

with the superpolynomial deformation:

$$M_\mu(q^{-\lambda}t^{-\rho}) = M_\mu\{q,t|\mathfrak{p}^{*\lambda}_k\}$$

Thus, the final result is

$$\boxed{\mathcal{P}^{\text{Hopf}}_{\lambda,\mu} = \mathcal{M}_\lambda \cdot M_\mu\{q,t|\mathfrak{p}^{*\lambda}_k\}}$$

Thank you for your attention!